

PERVERSE SHEAVES ON GRASSMANNIANS

TOM BRADEN

ABSTRACT. We compute the category of perverse sheaves on Hermitian symmetric spaces in types A and D, constructible with respect to the Schubert stratification. The calculation is microlocal, and uses the action of the Borel group to study the geometry of the conormal variety Λ .

CONTENTS

1. Preliminaries and statement of results	3
1.1. The classical (type A) Grassmannian	3
1.2. λ -pairs: type A	4
1.3. The quiver category: type A	5
1.4. The main result: type A	6
1.5. The type D isotropic Grassmannian	7
1.6. λ -pairs: type D	7
1.7. The main result: type D	8
1.8. Simple representations and finite dimensionality	9
2. Conormal geometry of the Grassmannian	11
2.1. Normal and conormal coordinates	11
2.2. Actions and orbits	13
2.3. Orbit structure of M_λ	14
2.4. Codimension zero orbits	17
2.5. Codimension one orbits	18
2.6. Codimension two orbits	18
2.7. Fundamental groups of orbits	19
2.8. Modifications for type D	20
3. Microlocal perverse sheaves	23
3.1. Stacks	23
3.2. The stack \mathcal{E}	24
3.3. Microlocal Fourier transform	25
3.4. Monodromic perverse sheaves	25
3.5. Normal crossings perverse sheaves	26
4. Proof of the main theorems	28
4.1. Reducing the theorems to microlocal geometry	28
4.2. Cutting by a normal slice	29

1991 *Mathematics Subject Classification*. Primary 32S60, Secondary 32C38, 35A27.

Key words and phrases. perverse sheaves, microlocal geometry.

This research supported in part by NSF grant DMS 9304580.

4.3. Codimension zero MPS	29
4.4. MPS through codimension one	30
4.5. Codimension two relations	32
4.6. Compatibility of orientations	32
4.7. Relating the functors F_λ	33
4.8. Proof of Proposition 4.5.1, continued	34
4.9. Proof of Proposition 4.5.1, concluded	36
References	37

In [2] Beilinson and Bernstein gave an equivalence of categories between certain categories of perverse sheaves on a flag variety G/B and certain representations of the Lie algebra \mathfrak{g} . This correspondence has most often been used to understand irreducible representations by studying the corresponding perverse sheaves. The complete structure of the category of perverse sheaves has been computed in a few simple cases (in particular, for projective spaces with the Schubert stratification) but existing techniques have not allowed computation of more complicated examples. In this paper we give a quiver description of the category $\mathcal{P}_\Lambda(X)$ when $X = G/P$ is a Hermitian symmetric space in type A or D (i.e. a Grassmannian in type A , or an isotropic Grassmannian in type D), stratified by the Schubert stratification. Here $\Lambda \subset T^*X$ is the conormal variety to the stratification; the category \mathcal{P}_Λ of perverse sheaves with characteristic variety contained in Λ is the same as the category of Schubert-constructible perverse sheaves.

Our strategy is to study perverse sheaves microlocally, that is, as objects supported on Λ . Such a description exists, at least theoretically, via the theory of regular singularities \mathcal{E} -modules. A topological description of this category was given in [11]. In practice, computing such a category proceeds from smooth points of Λ inwards to deeper singularities. A conjecture of Kashiwara says that only codimension 0, 1, and 2 pieces of Λ should be necessary in the computation; our methods show as a corollary that this conjecture holds for our spaces, although the proof is quite special to our particular geometry.

There are several pleasant features of our varieties which make the computation of $\mathcal{P}_\Lambda(X)$ reasonable, all of which fail for full flag varieties G/B . First, the action of Borel group B on the conormal variety Λ has finitely many orbits. This provides a natural stratification of Λ and allows a simple description of the geometry of the strata and how they intersect. The author is unaware of even an algorithm to decide whether two components of the conormal variety to the Schubert stratification of a full flag variety G/B meet in codimension one.

Second, the action of the Borel group B on Λ has connected stabilizers, so the fundamental groups of the orbits are free abelian groups, generated by $\pi_1(B) = \pi_1(T)$. Already for the full flag variety for SL_4 there are smooth components of Λ with nonabelian π_1 . If X is the type B Hermitian symmetric space (the Lagrangian Grassmannian), the stabilizers are not connected, so the fundamental

groups of the orbits cannot be completely described by the action of $\pi_1(T)$. This is essentially the reason we do not consider this case in this paper.

Finally, all the singularities of the Schubert stratification of G/P are conical. This simplifies things considerably; it means that all the codimension one intersections of components of Λ look like the conormal variety to a line bundle L stratified as $Z \cup (L \setminus Z)$ where Z is the zero section. It also allows the use of the Fourier transform to identify microlocal perverse sheaves on different spaces, rather than contact transformations, which are harder to compute with.

A general description of microlocal perverse sheaves on the union of codimension zero and one strata of Λ for conical stratifications was given in [6]; in that paper it was applied to stratifications where there are no codimension two strata. There are codimension two orbits for the spaces we consider, but the geometry of Λ near these orbits is as simple as possible. It is just the conormal variety to a direct sum of two line bundles $L_1 \oplus L_2$, with the “normal crossings” stratification.

The resulting presentation by generators and relations, while directly arising from the conormal geometry, is not algebraically the most pleasing. Mikhail Khovanov has described a quiver algebra [17] arising from an algebra of cobordisms, which he uses to “categorify” invariants of links and tangles. His algebra is isomorphic to a subquotient of our algebra for the type A Grassmannian; a proof will appear in [7]. Khovanov’s algebra is naturally graded, with quadratic relations, and thus our algebra is also, at least in the type A case. Such a grading is a crucial ingredient in Koszul duality theory [3].

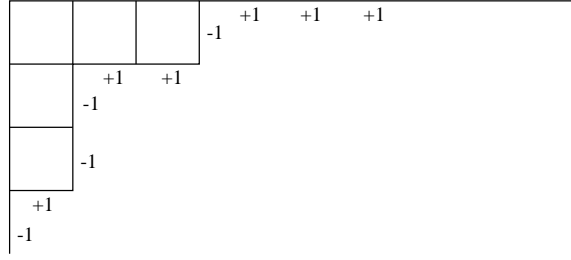
The paper is organized as follows. §1 presents the quiver category that describes the category $\mathcal{P}_\Lambda(X)$, and describes the simple objects in this category. After some combinatorial preliminaries the quiver categories are described in sections 1.4 (type A) and 1.6 (type D). §2 describes the geometry and combinatorics of the B -orbits of Λ . §3 introduces microlocal perverse sheaves, describes the building blocks (monodromic and normal crossings perverse sheaves) which are “glued together” to give the final answer, and finally §4 gives the necessary identifications to carry out the gluing.

1. PRELIMINARIES AND STATEMENT OF RESULTS

1.1. The classical (type A) Grassmannian. Consider the complex Grassmannian $X = X_{k,l}$ parametrizing k -dimensional sub-vector spaces of \mathbb{C}^n , where $n = k+l$. Letting $G = SL(n, \mathbb{C})$, then G acts on X transitively, and $X = G/P$ for $P = P_{k,l}$ a maximal parabolic subgroup of G . The Borel subgroup B of upper triangular matrices in G acts on X with finitely many orbits, called Schubert cells. These cells give a stratification of X which is described as follows.

Let Ω denote the collection of all partitions, i.e. nonincreasing sequences $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_i \geq \dots$ of nonnegative integers which are eventually zero. Let $\Omega_{k,l} \subset \Omega$ be the set of λ for which $\lambda_{k+1} = 0$ and $\lambda_i \leq l$ for all i . In other words, $\lambda \in \Omega_{k,l}$ if and only if the Young diagram $\Delta(\lambda)$ of λ fits in a rectangle with k rows and l columns. Here we put

$$\Delta(\lambda) = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}$$

FIGURE 1. Computing ϕ_λ by following the boundary of $\Delta(\lambda)$

(we are using matrix coordinates for $\mathbb{N} \times \mathbb{N}$: (i, j) is the point in the i th row and j th column from the upper left).

Denote the standard flag fixed by B by $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$, and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of \mathbb{C}^n . For $\lambda \in \Omega_{k,l}$ define the Schubert cell X_λ to be

$$X_\lambda = \{ V \in X_{k,l} \mid \dim(V \cap \mathbb{C}^{d_j}) = j, 1 \leq j \leq k \},$$

where we put $d_j = \lambda_{k-j+1} + j$. It is a smooth affine variety isomorphic to $\mathbb{C}^{|\lambda|}$, where $|\lambda| = \sum \lambda_i$. Put a partial order on Ω by defining $\lambda \leq \lambda'$ if and only if $\lambda_i \leq \lambda'_i$ for all i – i.e. by inclusion of Young diagrams. Then $X_\lambda \subset \overline{X_{\lambda'}}$ if and only if $\lambda \leq \lambda'$.

There is another way of describing a partition λ which will be useful. Let $\mathbb{H} = \mathbb{Z} + \frac{1}{2}$ be the set of “half integers”. Given λ , define a function $\phi = \phi_\lambda: \mathbb{H} \rightarrow \{+1, -1\}$ by letting $\phi(\alpha) = -1$ for $\alpha \in \{ \lambda_i - i + \frac{1}{2} \mid i > 0 \}$, and $\phi(\alpha) = +1$ otherwise. Geometrically, the function ϕ is obtained by moving along the outer edge of the Young diagram $\Delta(\lambda)$ and giving a -1 for every step up and a $+1$ for every step to the right. More precisely, $\phi(\alpha)$ is determined by the orientation of the boundary segment that intersects the line $y + x = \alpha$.

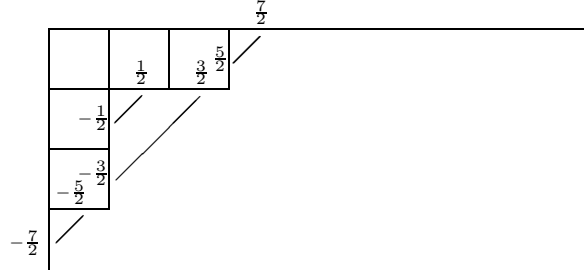
This produces a bijection between Ω and the set of all functions $\phi: \mathbb{H} \rightarrow \{+1, -1\}$ which are -1 for all sufficiently negative integers and $+1$ for all sufficiently positive ones, and for which the sum $\sum_{\alpha=-\beta}^{\beta} \phi(\alpha)$ is zero for all β large enough (See figure 1).

1.2. λ -pairs: type A. Let $\tilde{\Pi}(\lambda) = \phi^{-1}(-1) \times \phi^{-1}(+1) \subset \mathbb{H} \times \mathbb{H}$.

Definition. Call a pair $(\alpha, \beta) \in \tilde{\Pi}(\lambda)$ a “ λ -pair” if β is the smallest number for which $\beta > \alpha$ and $\sum_{\alpha \leq \gamma \leq \beta} \phi(\gamma) = 0$. Let $\Pi(\lambda) \subset \tilde{\Pi}(\lambda)$ be the set of λ -pairs.

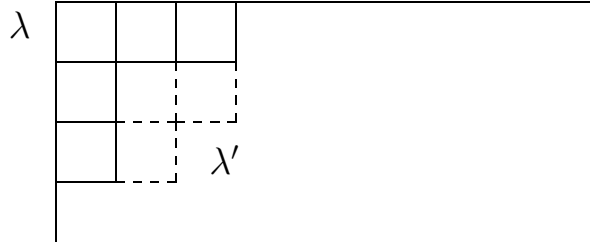
If $(\alpha, \beta) \in \Pi(\lambda)$, and we start from the center of the boundary segment of $\Delta(\lambda)$ corresponding to α , then β is the first boundary segment encountered by a ray extended up and to the right with slope $+1$. Figure 2 shows that for $\lambda = (3, 1, 1)$, $(-\frac{1}{2}, \frac{1}{2})$, $(-\frac{3}{2}, \frac{3}{2})$, $(-\frac{7}{2}, -\frac{5}{2})$, and $(\frac{5}{2}, \frac{7}{2})$ are all λ -pairs.

Lemma 1.2.1. *Every $\alpha \in \mathbb{H}$ appears in exactly one λ -pair. The λ -pairs define a set of nested intervals; if (α, β) and (α', β') are λ -pairs, and $\alpha' \in [\alpha, \beta]$, then $\beta' \in [\alpha, \beta]$ also.*


 FIGURE 2. Finding $\Pi([3, 1, 1])$

Put a partial order on $\tilde{\Pi}(\lambda)$ by letting $(\alpha, \beta) \leq (\alpha', \beta')$ whenever $\alpha' \leq \alpha$ and $\beta \leq \beta'$, and let $\Pi(\lambda)$ have the induced partial order. Lemma 1.2.1 implies that $\Pi(\lambda)$ is a tree; each element has a unique smallest element dominating it (its “parent”).

Given a λ -pair $\pi = (\alpha, \beta)$, define a new partition λ' by letting $\phi_{\lambda'}(\alpha) = +1$, $\phi_{\lambda'}(\beta) = -1$, and $\phi_{\lambda'}(\gamma) = \phi_{\lambda}(\gamma)$ otherwise (see Figure 3). We will denote this relation by $\lambda \xrightarrow{\pi} \lambda'$; if it holds for some λ -pair, we write $\lambda \rightarrow \lambda'$, and if $\lambda \rightarrow \lambda'$ or $\lambda' \rightarrow \lambda$, we write $\lambda \leftrightarrow \lambda'$.


 FIGURE 3. $\lambda \xrightarrow{\pi} \lambda'$, $\pi = (-\frac{3}{2}, \frac{3}{2})$

Let $\Pi_{k,l}(\lambda)$ be the subset of pairs $(\alpha, \beta) \in \Pi(\lambda)$ for which $\alpha > -k$ and $\beta < l$.

Lemma 1.2.2. *Suppose that $\lambda \xrightarrow{\pi} \lambda'$, and $\lambda \in \Omega_{k,l}$. Then $\lambda' \in \Omega_{k,l}$ if and only if $\pi \in \Pi_{k,l}(\lambda)$.*

Our interest in this relation comes because $\lambda \leftrightarrow \lambda'$ if and only if the corresponding irreducible components $\overline{T_{X_\lambda}^* X}$ and $\overline{T_{X_{\lambda'}}^* X}$ of the conormal variety Λ intersect in codimension one; we prove this in §2 (see Corollary 2.5.2). Lascoux and Schutzenberger studied this relation in [18]; they showed that $\lambda \rightarrow \lambda'$ if and only if the stalk intersection cohomology group

$$IH_x^{|\lambda'| - |\lambda| - 1}(\overline{X_{\lambda'}})$$

is nonzero, where x is any point of X_λ .

1.3. The quiver category: type A. In this section, we define the quiver category that describes perverse sheaves on the Grassmannians $X_{k,l}$.

Definition. A *diamond* is a 4-tuple $(\lambda, \lambda', \lambda'', \lambda''')$ of distinct elements in Ω satisfying $\lambda \leftrightarrow \lambda' \leftrightarrow \lambda'' \leftrightarrow \lambda''' \leftrightarrow \lambda$.

Let $\mathcal{A}_{k,l}$ be the category whose objects are collections of \mathbb{Q} -vector spaces V_λ , $\lambda \in \Omega_{k,l}$, together with two types of homomorphisms:

- maps t_α for each $\alpha \in \mathbb{H}$, acting on $V = \bigoplus V_\lambda$, and preserving this decomposition, and
- maps $p(\lambda, \lambda'): V_{\lambda'} \rightarrow V_\lambda$ for every pair $\lambda, \lambda' \in \Omega_{k,l}$ with $\lambda \leftrightarrow \lambda'$.

If $\lambda \leftrightarrow \lambda'$, define $\mu(\lambda, \lambda') = 1 + p(\lambda, \lambda')p(\lambda', \lambda)$. These homomorphisms are required to satisfy the following relations:

1. The maps t_α commute with each other and with the p 's.
2. If (α, β) is a λ -pair, then $t_\alpha t_\beta|_{V_\lambda} = 1|_{V_\lambda}$. If $\alpha < -k$ or $\alpha > l$, then $t_\alpha = 1$.
3. Suppose that $\lambda \xrightarrow{(\alpha, \beta)} \lambda'$, and (α', β') is the parent of (α, β) in $\Pi(\lambda)$. Then

$$\begin{aligned} \mu(\lambda', \lambda)^{\eta(\beta)} &= t_\alpha t_{\beta'}|_{V_{\lambda'}}, \text{ and} \\ \mu(\lambda, \lambda')^{\eta(\beta)} &= t_\alpha t_{\beta'}|_{V_\lambda}, \end{aligned}$$

where $\eta(\beta) = (-1)^{\beta + \frac{1}{2}}$.

4. If $\lambda, \lambda', \lambda'',$ and $\lambda''' \in \Omega_{k,l}$ form a diamond, then

$$p(\lambda'', \lambda')p(\lambda', \lambda) = p(\lambda'', \lambda''')p(\lambda''', \lambda).$$

If we have a diamond $(\lambda, \lambda', \lambda'', \lambda''')$, and all elements except λ''' are in $\Omega_{k,l}$, then

$$p(\lambda, \lambda')p(\lambda', \lambda'') = 0 = p(\lambda'', \lambda')p(\lambda', \lambda).$$

Maps between objects of $\mathcal{A}_{k,l}$ are collections of maps $V_\lambda \rightarrow V_{\lambda'}$ which are compatible with the t 's and p 's. Note that the presentation above admits simplifications; for instance, Lemma 1.8.1 below shows that the t_α can be expressed in terms of the p 's.

Objects of $\mathcal{A}_{k,l}$ are finite dimensional representations of the \mathbb{Q} -algebra $A_{k,l}$ generated by the t_α , $p(\lambda, \lambda')$, and commuting idempotents $e_{\lambda'}$ representing the projections $\bigoplus V_\lambda \rightarrow V_{\lambda'}$, modulo the relations above.

Proposition 1.3.1. *The algebra $A_{k,l}$ is finite dimensional over \mathbb{Q} .*

Proof. This follows from Proposition 1.8.4 below. \square

1.4. The main result: type A. In §4 we will define a functor $R: \mathcal{P}_\Lambda(X_{k,l}) \rightarrow \mathcal{A}_{k,l}$ from Schubert-constructible perverse sheaves to quiver representations. If $R(\mathbf{P}) = \{V_\lambda, t_\alpha, p(\lambda, \lambda')\}$, then V_λ is the vanishing cycles group, or Morse group, of \mathbf{P} at a point in the Schubert stratum X_λ corresponding to λ .

Theorem 1.4.1. *R is an equivalence of categories.*

The form of the quiver algebra $\mathcal{A}_{k,l}$ reflects the geometry of the conormal variety Λ associated to the Schubert stratification of $X_{k,l}$. The action of t_α represents the monodromy of the vanishing cycles local systems (which are local systems on the smooth components of Λ) around loops generated by the action of a loop $\gamma_\alpha \in \pi_1(B)$.

The quiver relation (2) describes the kernel of the homomorphisms $\pi_1(B) \rightarrow \pi_1(\tilde{\Lambda}_{X_\lambda})$, where $\tilde{\Lambda}_{X_\lambda}$ is the smooth part of the conormal variety lying over X_λ . (3) relates the monodromies of loops in $\tilde{\Lambda}_{X_\lambda}$ and $\tilde{\Lambda}_{X_{\lambda'}}$ around the intersection of their closures; as we shall see in §2.5, they intersect in a divisor if and only if $\lambda \rightarrow \lambda'$. Finally, (4) comes from the codimension two strata of Λ ; it is analogous to, and follows from, a similar relation in the quiver for perverse sheaves on \mathbb{C}^2 stratified by normal crossings.

1.5. The type D isotropic Grassmannian. To describe the other space we will consider, let $n = 2k$, and take a nondegenerate quadratic form on \mathbb{C}^n :

$$Q(x_1, \dots, x_{2k}) = x_1x_{2k} + x_2x_{2k-1} + \dots + x_kx_{k+1}.$$

The space $\{V \in X_{k,k} \mid Q|_V = 0\}$ has two connected components; let X_k^s be the component containing the point $\{x_{k+1}, \dots, x_n = 0\}$. The group $G^s = SO(n, \mathbb{C}) \subset SL(n, \mathbb{C})$ of transformations preserving Q acts on X_k^s , and $X_k^s = G^s / (G^s \cap P_{k,k})$. Throughout this article, we will refer to this space as the “type D case”, and we will use a superscript “s” to distinguish structures used in this case.

The group $B^s = G^s \cap B$ is a Borel subgroup of G^s . The Schubert stratification of X_k^s is the stratification by B^s orbits, which are just the spaces $X_\lambda^s = X_\lambda \cap X_k^s$. Let $\Omega^s \subset \Omega$ be the set of partitions whose Young diagrams are symmetric about the diagonal and have an even number of squares on the diagonal. We have

$$\Omega^s = \{\lambda \in \Omega \mid \phi_\lambda(\alpha) = -\phi_\lambda(-\alpha) \text{ for all } \alpha \in \mathbb{H} \text{ and } |\lambda| \text{ is even}\}.$$

Let $\Omega_k^s = \Omega^s \cap \Omega_{k,k}$; then $\lambda \in \Omega_k^s$ if and only if $X_\lambda^s \neq \emptyset$.

1.6. λ -pairs: type D. Again we define a set $\Pi^s(\lambda) \subset \tilde{\Pi}(\lambda)$ of λ -pairs for $\lambda \in \Omega^s$.

Definition. If $(\alpha, \beta) \in \tilde{\Pi}(\lambda)$, we let $(\alpha, \beta) \in \Pi^s(\lambda)$ if one of the following two conditions holds:

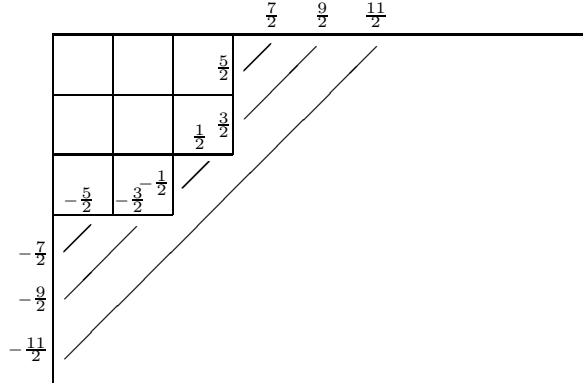
- $\alpha > 0$, $\beta > 0$, and $(\alpha, \beta) \in \Pi(\lambda)$, or
- $\beta > -\alpha > 0$, $(\alpha, -\alpha)$ and $(-\beta, \beta)$ are both in $\Pi(\lambda)$, $\alpha + \frac{1}{2}$ is even, and $\sum_{-\alpha < \gamma \leq \beta} \phi_\lambda(\gamma) = +1$.

(The last equation simply says that $(-\beta, \beta)$ is the parent of $(\alpha, -\alpha)$ in $\Pi(\lambda)$.) Let $\Pi_k^s(\lambda) \subset \Pi^s(\lambda)$ be the subset of (α, β) for which $\beta < k$. Give $\Pi^s(\lambda)$ the partial ordering induced from $\tilde{\Pi}(\lambda)$.

Lemma 1.6.1. *For any $\alpha \in \mathbb{H}$ there is exactly one type D λ -pair in which either α or $-\alpha$ appears. $\Pi^s(\lambda)$ is a tree with the given partial ordering.*

Proof. Since the Young diagram of λ is symmetric, we have $(\alpha, \beta) \in \Pi(\lambda)$ if and only if $(-\beta, -\alpha) \in \Pi(\lambda)$. Elements (α, β) where α, β have the same sign thus come in pairs; the first part of the definition of $\Pi^s(\lambda)$ just takes the one with positive α and β .

The remaining elements of $\Pi(\lambda)$ must be of the form $(-\alpha, \alpha)$ for $\alpha > 0$. These form a sequence of nested intervals $(\alpha_1, -\alpha_1) < (\alpha_2, -\alpha_2) < \dots$. Since each of the sets $[0, -\alpha_1) \cap \mathbb{H}$ and $(-\alpha_i, -\alpha_{i+1}) \cap \mathbb{H}$ must be a union of pairs in $\Pi(\lambda)$, we see that $\alpha_i + \frac{1}{2} + i$ is even for all i .

FIGURE 4. Finding $\Pi^s([3, 3, 2])$.

The second part of the definition above thus puts $(\alpha_{2i-1}, -\alpha_{2i})$ in $\Pi^s(\lambda)$ for all $i \in \mathbb{N}$. Since any α appears in exactly one pair in $\Pi(\lambda)$, the first statement follows.

The fact that $\Pi^s(\lambda)$ is a tree now follows easily from the fact that $\Pi(\lambda)$ is a tree. \square

To give an example, let $\lambda = (3, 3, 2)$. Figure 4 shows that $(\frac{5}{2}, \frac{7}{2})$, $(\frac{3}{2}, \frac{9}{2})$ and $(-\frac{1}{2}, \frac{11}{2})$ are all in $\Pi^s(\lambda)$.

If $(\alpha, \beta) \in \Pi^s(\lambda)$, define a second partition $\lambda' \in \Omega$ by switching the signs of $\phi_\lambda(\alpha)$, $\phi_\lambda(\beta)$, $\phi_\lambda(-\alpha)$, and $\phi_\lambda(-\beta)$. It is easy to check that $\lambda' \in \Omega^s$. As before, we write $\lambda \xrightarrow{(\alpha, \beta)} \lambda'$ to express this relation.

1.7. The main result: type D. Now we define a quiver category \mathcal{A}_k^s analogous to the category $\mathcal{A}_{k,l}$ from the last section. Objects are collections of finite dimensional vector spaces V_λ , one for each $\lambda \in \Omega_k^s$, together with maps t_α for $\alpha \in \mathbb{H}$ and $p(\lambda, \lambda')$ for pairs $\lambda \leftrightarrow \lambda'$, as before.

Once again we define $\mu(\lambda, \lambda') = 1 + p(\lambda, \lambda')p(\lambda', \lambda)$ whenever $\lambda \leftrightarrow \lambda'$. These maps are then required to satisfy the following relations, plus the relation (1) from the definition of $\mathcal{A}_{k,l}$:

- (2s) $t_\alpha t_{-\alpha} = 1$ for all α , and $t_\alpha = 1$ if $\alpha > k$. If (α, β) is a λ -pair, then $t_\alpha t_\beta|_{V_\lambda} = 1|_{V_\lambda}$.
- (3s) Suppose that $\lambda \xrightarrow{(\alpha, \beta)} \lambda'$, and $(\alpha', \beta') \in \Pi^s(\lambda)$ is the parent of (α, β) . If $(\alpha, \beta) < (-\beta', -\alpha')$, then let $\zeta = -\alpha'$; otherwise let $\zeta = \beta'$. Then

$$\begin{aligned} \mu(\lambda', \lambda)^{\eta(\beta)} &= t_\alpha t_\zeta|_{V_{\lambda'}}, \text{ and} \\ \mu(\lambda, \lambda')^{\eta(\beta)} &= t_\alpha t_\zeta|_{V_\lambda}, \end{aligned}$$

where $\eta(\beta) = (-1)^{\beta + \frac{1}{2}}$. (Note that because of relation 2s, the second equation can also be written $\mu(\lambda, \lambda')^{\eta(\beta)} = t_\alpha t_{\beta'}|_{V_\lambda}$.)

- (4s) The first sentence of relation (4) from $\mathcal{A}_{k,l}$ holds, but the second part is modified as follows: If we have a triple $\lambda \leftrightarrow \lambda' \leftrightarrow \lambda''$ in Ω_k^s for which either (a)

there is a diamond $(\lambda, \lambda', \lambda'', \lambda''')$, where $\lambda''' \in \Omega^s \setminus \Omega_k^s$ or (b) $\lambda \xrightarrow{\pi} \lambda' \xrightarrow{\pi'} \lambda''$ with $\pi = (\alpha, \beta)$ a λ -pair with $\alpha < 0$ and $\pi' \notin \Pi^s(\lambda)$, then

$$p(\lambda, \lambda')p(\lambda', \lambda'') = 0 = p(\lambda'', \lambda')p(\lambda', \lambda).$$

Here the definition of a diamond is the same as in §1.3: a 4-tuple $(\lambda, \lambda', \lambda'', \lambda''')$ of distinct elements in Ω^s satisfying $\lambda \leftrightarrow \lambda' \leftrightarrow \lambda'' \leftrightarrow \lambda''' \leftrightarrow \lambda$.

Define an algebra A_k^s in the same way as before, so that \mathcal{A}_k^s is the category of finite dimensional representations of A_k^s .

Proposition 1.7.1. *The algebra A_k^s is finite dimensional over \mathbb{Q} .*

Proof. This follows from Proposition 1.8.4 below. \square

We will define a functor $R: \mathcal{P}_\Lambda(X_k^s) \rightarrow \mathcal{A}_k^s$ just as in the type A case. Our main result for the type D Grassmannian is:

Theorem 1.7.2. *R is an equivalence of categories.*

The definition of R and the proof of Theorem 1.7.2 will be given in §4.

1.8. Simple representations and finite dimensionality. In this section we prove that the algebras $A = A_{k,l}$ and $A = A_k^s$ are finite dimensional, and describe their irreducible representations.

Lemma 1.8.1. *Take $\gamma \in \mathbb{H}$, and assume that $\gamma > 0$ in the type D case. Then we have*

$$(t_\gamma|_{V_\lambda})^{\phi_\lambda(\gamma)} = \prod \mu(\lambda, \lambda')^{\eta(\beta)},$$

where the product is over all $\lambda' \in \Omega_{k,l}$ (resp. Ω_k^s) for which $\lambda \xrightarrow{(\alpha, \beta)} \lambda'$ for some λ -pair with $\alpha \leq \gamma \leq \beta$.

Proof. Using the quiver relations 2 and 3 (or 2s and 3s), we see that if $\lambda \xrightarrow{(\alpha, \beta)} \lambda'$, and (α', β') is the parent of (α, β) we have $\mu(\lambda, \lambda')^{\eta(\beta)} = t_\alpha t_{\alpha'}^{-1}$ on V_λ . The product above is thus a telescoping product, since $t_\alpha = 1$ for all sufficiently negative α . The telescope starts with $t_\gamma^{\pm 1}$, where the sign is determined by $\phi_\lambda(\gamma)$, i.e. by whether γ appears first or second in a λ -pair. \square

Proposition 1.8.2. *If $\lambda' \rightarrow \lambda$ then $\mu(\lambda, \lambda')$ and $\mu(\lambda', \lambda)$ are both unipotent in A .*

Proof. First note that $\mu(\lambda, \lambda')$ is unipotent if and only if $\mu(\lambda', \lambda)$ is, since they are of the form $1 + pq$ and $1 + qp$, respectively.

We proceed by downward induction in λ . If λ is the maximal element in $\Omega_{k,l}$ (or Ω_k^s), then the quiver relation 3 implies $\mu(\lambda, \lambda') = 1$, since all t_α act as the identity on V_λ .

Now suppose the proposition holds for all $\lambda > \lambda_0$, and take some $\lambda' \rightarrow \lambda_0$. Lemma 1.8.1 and the quiver condition 3 can be used to express $\mu(\lambda_0, \lambda')$ as a product of powers (positive and negative) of $\mu(\lambda_0, \lambda'')$ for $\lambda_0 \rightarrow \lambda''$; these are unipotent by the inductive assumption, and they commute by quiver relation 1. The unipotence of $\mu(\lambda_0, \lambda')$ and $\mu(\lambda', \lambda_0)$ now follow. \square

Lemma 1.8.3. *Suppose that $\lambda \rightarrow \lambda_1$, $\lambda \rightarrow \lambda_2$ and $\lambda_1 \neq \lambda_2$ (in either Ω or Ω^s). Then we can complete this arrangement to a diamond, i.e. there exists a $\lambda' \in \Omega$ (resp. Ω^s) with $\lambda' \neq \lambda$ for which $\lambda_1 \leftrightarrow \lambda' \leftrightarrow \lambda_2$. Furthermore, for any such λ' , either $\lambda' > \lambda_1$ or $\lambda' > \lambda_2$.*

Proof. Consider the case of Ω ; the argument for Ω^s is similar and we will omit it. Suppose that $\lambda \xrightarrow{\pi_1} \lambda_1$ and $\lambda \xrightarrow{\pi_2} \lambda_2$. Recall the tree structure on the set $\Pi(\lambda)$ of λ -pairs. If neither π_1 or π_2 is the parent of the other, then we have $\pi_1 \in \Pi(\lambda_2)$ and $\pi_2 \in \Pi(\lambda_1)$, and there is a λ' with $\lambda_1 \xrightarrow{\pi_2} \lambda'$ and $\lambda_2 \xrightarrow{\pi_1} \lambda'$.

On the other hand, if $\pi_2 = (\alpha_2, \beta_2)$ is the parent of $\pi_1 = (\alpha_1, \beta_1)$, then (α_2, α_1) and (β_1, β_2) are in $\Pi(\lambda_1)$, and we get λ_l, λ_r for which $\lambda_1 \xrightarrow{(\alpha_2, \alpha_1)} \lambda_l \xrightarrow{(\beta_1, \beta_2)} \lambda_2$ and $\lambda_1 \xrightarrow{(\beta_1, \beta_2)} \lambda_r \xrightarrow{(\alpha_2, \alpha_1)} \lambda_2$.

For the last statement, just check that in both cases above we have found the only possible $\lambda' \neq \lambda$ for which $\lambda_1 \leftrightarrow \lambda' \leftrightarrow \lambda_2$. \square

Since $p(\lambda, \lambda')p(\lambda'', \lambda''') = 0$ unless $\lambda' = \lambda''$, a nonzero monomial in the p 's can be seen as a path in the graph Γ whose nodes are elements of $\Omega_{k,l}$ (or Ω_k^s), with edges are given by the relation “ \leftrightarrow ”. Let the idempotent e_λ be the monomial corresponding to the trivial path with only one node λ . Let $(\lambda_1, \lambda_2, \dots, \lambda_j)$ be a path in Γ . We will say that λ_i is a *valley* if $\lambda_{i-1} > \lambda_i$ and $\lambda_{i+1} > \lambda_i$.

Proposition 1.8.4. *The algebra A is spanned as a vector space over \mathbb{C} by the monomials without valleys. A monomial corresponding to a path of length l is a \mathbb{Z} -linear combination of monomials without valleys, all coming from paths of length $\geq l$.*

Proof. First, note that as a consequence of Lemma 1.8.1, we only need to show that monomials in the p 's can be expressed in terms of monomials without valleys.

Consider the monomial m with path $(\lambda_1, \lambda_2, \dots, \lambda_j)$, and let λ_i be a valley. If $\lambda_{i-1} \neq \lambda_{i+1}$, then we can apply Lemma 1.8.3 to obtain another expression for m as a monomial of the same length without a valley in the i th place.

If $\lambda_{i-1} = \lambda_{i+1}$, then we have $p(\lambda_{i-1}, \lambda_i)p(\lambda_i, \lambda_{i-1}) = \mu(\lambda_{i-1}, \lambda_i) - 1$. Now apply Lemma 1.8.1 to express $\mu(\lambda_{i-1}, \lambda_i)$ as a product of terms $\mu(\lambda_{i-1}, \lambda)^{\pm 1}$ for $\lambda > \lambda_{i-1}$. Since all these terms are unipotent, we can use the substitution $(1 + pq)^{-1} = \sum (-pq)^k$ to get an expression for $\mu(\lambda_{i-1}, \lambda_i)$ as a linear combination of monomials with length ≥ 2 and which only visit nodes $\lambda \geq \lambda_{i-1}$.

In both cases, the only new valleys created are above λ_i . So we can apply this process repeatedly, first getting rid of all valleys for which $|\lambda_i| = 0$, then for $|\lambda_i| = 1$, and so on. This process will terminate, since any monomials whose paths go outside $\Omega_{k,l}$ (resp. Ω_k^s) are zero. \square

The finite dimensionality of $A_{k,l}$ and A_k^s follows: any path without valleys is a composition of an increasing path with a decreasing path, and there are clearly only finitely many of these. As a further consequence, we obtain the following description of the irreducible representations of the algebras $A_{k,l}$ and A_k^s , which we will need in the final step of our proof of theorems 1.4.1 and 1.7.2.

Theorem 1.8.5. *All irreducible representations of $A_{k,l}$ and A_k^s are one dimensional. They are in one-to-one correspondence with elements $\Omega_{k,l}$ (resp. Ω_k^s).*

Proof. Suppose $V = \{V_\lambda, t_\alpha, p(\lambda, \lambda')\}$ is an irreducible representation. Choose a nonzero vector $v \in V_\lambda$. Let $(\lambda_1, \dots, \lambda_j)$ be the longest path for which the corresponding monomial m acts nontrivially on v ; one exists because of Proposition 1.8.4. Then for any $\lambda \leftrightarrow \lambda_j$, we have $p(\lambda, \lambda_j)m \cdot v = 0$. If $V' = \{V'_\lambda, t'_\alpha, p'(\lambda, \lambda')\}$ is the irreducible representation for which V'_{λ_j} is one dimensional and all other V'_λ vanish, then there is a map $V' \rightarrow V$ given by sending a generator of V'_{λ_j} to $m \cdot v$. This contradicts the irreducibility of V unless the path was trivial and $V' = V$. \square

Remark. Combining Theorems 1.8.5, 1.4.1, and 1.7.2, we see that simple perverse sheaves in $\mathcal{P}_\Lambda(X_{k,l})$ and $\mathcal{P}_\Lambda(X_k^s)$ have nonzero vanishing cycle groups at only one stratum. This was proved in the type A case by Bressler, Finkelberg and Lunts [8], and in the type D case by Boe and Fu [4].

2. CONORMAL GEOMETRY OF THE GRASSMANNIAN

In this section we study the geometry of the conormal variety Λ to the Schubert stratifications of $X = X_{k,l}$ and $X = X_k^s$. The key facts, in both cases, are:

- The Borel group B acts on Λ with finitely many orbits.
- The stabilizers of the B -action on Λ are connected, so the fundamental groups of orbits are quotients of $\pi_1(B)$.
- Near a point in a codimension one or two orbit Λ has only normal crossings singularities.

More precisely we focus on the fiber M_λ of Λ over a torus-fixed point W_λ in X_λ . The stabilizer $B_\lambda = B_{W_\lambda}$ acts on this fiber with finitely many orbits (Proposition 2.2.2). The B -orbits of Λ lying over X_λ are isomorphic to $X_\lambda \times O$, for $O \subset M_\lambda$ a B_λ -orbit.

We begin with the case $X = X_{k,l}$, and finish with an outline of the differences in the type D case.

2.1. Normal and conormal coordinates. We first introduce the coordinate systems we will use to describe the geometry of X and $\Lambda \subset T^*X$.

Given a partition $\lambda \in \Omega_{k,l}$, let W_λ be the unique point of X_λ which is fixed by the torus $T \subset B$ of diagonal matrices. More explicitly, W_λ is spanned by $\{\mathbf{e}_i \mid i \in I\}$, where

$$\begin{aligned} I &= \{ \lambda_k + 1, \lambda_{k-1} + 2, \dots, \lambda_1 + k \} \\ &= \{ \alpha + k + \tfrac{1}{2} \mid \phi_\lambda(\alpha) = -1 \text{ and } -k < \alpha < l \}. \end{aligned}$$

Let $I' = \{1, \dots, n\} \setminus I$. For the rest of this section the partition λ will be fixed and we will put $W = W_\lambda$.

Let $G = SL(n, \mathbb{C})$, and let $P = G_W$ be the stabilizer of W , with Lie algebra \mathfrak{p} . The infinitesimal action of \mathfrak{g} on X induces an isomorphism $T_W X \cong \mathfrak{g}/\mathfrak{p}$. Since $\mathfrak{p} = \{g \in \mathfrak{g} \mid g(W) \subset W\}$, we get identifications $T_W X \cong \text{Hom}(W, \mathbb{C}^n/W)$ and $T_W^* X \cong \text{Hom}(\mathbb{C}^n/W, W)$.

We represent elements of $\text{Hom}(\mathbb{C}^n/W, W)$ by $k \times l$ matrices, where we take $\{\mathbf{e}_i\}_{i \in I'}$, in order of *decreasing* i , as a basis for W , and the residues of $\{\mathbf{e}_i\}_{i \in I'}$, taken in increasing order, as a basis \mathbb{C}^n/W . Let R be the rectangle $\{1, \dots, k\} \times \{1, \dots, l\}$.

For $A \in \text{Hom}(\mathbb{C}^n/W, W)$, define the *support* $\text{supp } A \subset R$ of A to be the set of $(i, j) \in R$ such that $A_{ij} \neq 0$. For a subset $S \subset R$, let $\mathbb{C}^S = \{A \mid \text{supp } A \subset S\}$. We use the usual convention for matrix coordinates so (i, j) denotes a point in the i th row from the top and the j th column from the left. We will abuse notation slightly and represent elements of $\text{Hom}(W, \mathbb{C}^n/W)$ by $k \times l$ matrices also, using the duality given by the basis of elementary matrices.

Let $M_\lambda = T_W^* X \cap T_{X_\lambda}^* X$, the fiber of Λ over W . Put $R_\lambda = R \setminus \Delta(\lambda)$, where $\Delta(\lambda)$ is the Young diagram of λ , as introduced in §1.1.

Lemma 2.1.1. *With the above conventions, $M_\lambda = \mathbb{C}^{R_\lambda}$.*

Before proving this, we need to define a map relating the (i, j) matrix coordinates with coordinates in \mathbb{C}^n . Let $w_\lambda: \tilde{\Pi}(\lambda) \rightarrow \mathbb{N} \times \mathbb{N}$ be given by $w_\lambda(\alpha, \beta) = (w_1(\alpha), w_2(\beta))$, where

$$w_1(\alpha) = \#\{\alpha' \in \mathbb{H} \mid \alpha' \geq \alpha \text{ and } \phi_\lambda(\alpha') = -1\}, \text{ and}$$

$$w_2(\beta) = \#\{\beta' \in \mathbb{H} \mid \beta' \leq \beta \text{ and } \phi_\lambda(\beta') = +1\}.$$

Most of the time, the element λ will be fixed and we will drop the subscript λ . In terms of the Young diagram pictures, $w(\alpha, \beta)$ is the point in the same row as the vertical segment on the boundary of $\Delta(\lambda)$ indexed by α and the same column as the horizontal segment indexed by β .

Proof of Lemma 2.1.1. It is enough to show that $T_W X_\lambda = \mathbb{C}^{\Delta(\lambda)}$. Clearly $T_W X_\lambda$ is the image of the map $\mathfrak{b} \rightarrow \mathfrak{g}/\mathfrak{p} \cong \text{Hom}(W, \mathbb{C}^n/W)$. Diagonal elements of \mathfrak{b} are in the kernel, and the elementary matrix E_{ij} ($i \neq j$) is in \mathfrak{p} unless $i \in I'$, $j \in I$, in which case it maps to the elementary matrix with coordinate $w(j - k - \frac{1}{2}, i - k - \frac{1}{2})$ in $\text{Hom}(W, \mathbb{C}^n/W)$. The result now follows from the following lemma. \square

Lemma 2.1.2. *If $(\alpha, \beta) \in \tilde{\Pi}(\lambda)$, we have $w(\alpha, \beta) \in \Delta(\lambda)$ if and only if $\alpha > \beta$.*

Next define a map $\tilde{\epsilon} = \tilde{\epsilon}_\lambda: \text{Hom}(W, \mathbb{C}^n/W) \rightarrow X$ by identifying \mathbb{C}^n with $W \oplus \mathbb{C}^n/W$ using the basis $\{\mathbf{e}_i\}$ and letting $\tilde{\epsilon}(A)$ be the graph of A . This embeds $T_W X$ as a tubular neighborhood of X_λ .

Let $\bar{M}_\lambda = \mathbb{C}^{\Delta(\lambda)}$, so we have a splitting $\text{Hom}(W, \mathbb{C}^n/W) = M_\lambda \oplus \bar{M}_\lambda$. This gives an inclusion $M_\lambda^* \subset \text{Hom}(W, \mathbb{C}^n/W)^* = \text{Hom}(\mathbb{C}^n/W, W)$. Let $\epsilon: M_\lambda^* \rightarrow X$ be the restriction of $\tilde{\epsilon}$. The next result shows that ϵ is the inclusion of a normal slice to X_λ , and the stratification in $\tilde{\epsilon}(T_W X)$ is the product of X_λ with the stratification in the normal slice.

Proposition 2.1.3. *If $q: \text{Hom}(W, \mathbb{C}^n/W) \rightarrow M_\lambda^*$ is the projection map, then $\tilde{\epsilon}^{-1}(X_{\lambda'}) = q^{-1}\epsilon^{-1}(X_{\lambda'})$ for any $\lambda' \geq \lambda$.*

Proof. Suppose $A \in \text{Hom}(W, \mathbb{C}^n/W)$ has $A_{xy} = c \neq 0$ for some $(x, y) \in \Delta(\lambda)$. If $(i - k - \frac{1}{2}, j - k - \frac{1}{2}) = w^{-1}(x, y)$, then $\mathbf{e}_i, \mathbf{e}_j$ are the standard basis elements corresponding to the column and row of the square (x, y) . By Lemma 2.1.2, we have $i > j$. Then acting on $\tilde{\epsilon}(A)$ by the matrix $I - cE_{ji} \in B$ kills the entry at (x, y) . Repeating this argument shows that $\tilde{\epsilon}(A)$ and $\epsilon \circ q(A)$ always lie in the same Schubert cell. \square

2.2. Actions and orbits. Keeping the notation $W = W_\lambda$, let $B_\lambda = B_W \subset B$ be the stabilizer of W ; it acts on T^*X fixing T_W^*X and Λ , so it acts on M_λ . To describe this action, take an element $g \in B_\lambda$. It induces endomorphisms g_1 and g_2 on W and on \mathbb{C}/W . If $A \in M_\lambda$, then $g \cdot A = g_1 A g_2^{-1}$. In other words, the action is generated by the elementary row and column operations, where row i can only be added to row i' for $i < i'$ and column j to column j' for $j < j'$.

The action of B_λ on the dual space M_λ^* can be expressed similarly, in terms of “truncated row and column operations”: a row can be added to a row above and a column to a column to the left, but anything appearing inside $\Delta(\lambda)$ must be discarded. Note that both these actions include multiplication by scalars, so all the orbits are conical.

Proposition 2.2.1. *There is a one-to-one correspondence sending B -orbits $\mathcal{O} \subset \Lambda$ to pairs (λ, O) where $\lambda \in \Omega_{k,l}$ (or Ω_k^s) and O is a B_λ orbit in M_λ . It is given by choosing λ so that \mathcal{O} lies over the Schubert cell X_λ , and letting $O = M_\lambda \cap \mathcal{O}$.*

We will call a matrix A a “0-1 matrix” if all its entries A_{ij} are 0 or 1, and each row and column has at most one nonzero entry. The following proposition follows easily from the corresponding result for square matrices.

Proposition 2.2.2. *B_λ acts with finitely many orbits on both M_λ and M_λ^* . In both cases, any orbit contains a unique point given by a 0-1 matrix. The orbit a matrix A belongs to determines and is determined by the ranks of the submatrices A_ρ , where $\rho \subset R$ runs over all rectangles $\rho \subset R_{k,l}$ with $(1,1) \in \rho$ (for M_λ) and rectangles $\rho \subset R_\lambda$ with $(k,l) \in \rho$ (for M_λ^*). If A and A' are 0-1 matrices, $B_\lambda A' \subset \overline{B_\lambda A} \iff \text{rank } A'_\rho \leq \text{rank } A_\rho$ for all ρ .*

Note that $\text{rank } A_\rho = \#(\rho \cap \text{supp}(A))$ if A is a 0-1 matrix.

Corollary 2.2.3. *The orbit stratifications of M_λ and M_λ^* are dual, i.e. there is a bijective correspondence $O \rightarrow O^*$ from the orbits of M_λ to the orbits of M_λ^* so that the closure \overline{O} is the dual cone to $\overline{O^*}$.*

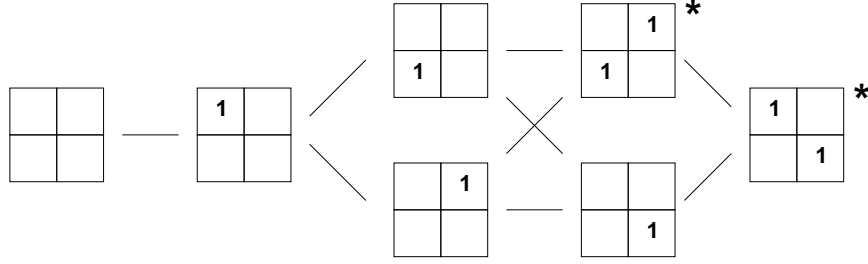
Proof. The dual cone to \overline{O} is a B_λ -invariant irreducible variety, and hence the closure of an orbit O^* . \square

The maps $\tilde{\epsilon}$ and ϵ are not B_λ -equivariant, but they are equivariant under the maximal torus in B_λ . We have the following result:

Proposition 2.2.4. *For every λ the set $\epsilon^{-1}(X_\lambda)$ is a union of B_λ -orbits.*

Proof. In [4], Boe and Fu show that the $\epsilon^{-1}(\overline{X_\lambda})$ are cut out by conditions on the ranks of submatrices containing (k,l) and touching, but not crossing, the boundary of $\Delta(\lambda)$. Since the orbits are determined by the ranks of all rectangular submatrices containing (k,l) , the orbit decomposition is finer. \square

In general the two decompositions of M_λ^* are not the same. For instance, take $k = l = 2$, $\lambda = \emptyset$ the zero partition. Figure 5 shows the 0-1 matrices of the orbits of M_λ^* (omitting the zeros). The lines give the codimension one closure relations, with the larger orbit placed to the right of the smaller one. Each $\epsilon^{-1}(X_\lambda)$ consists of a

FIGURE 5. The orbit structure of M_λ^* , $k = l = 2$, $\lambda = \emptyset$.

single B_λ orbit, except for $\lambda = (2, 2)$, when it is the union of the two orbits labeled with asterisks.

The following lemma characterizes which B_λ orbits do correspond to Schubert cells.

Lemma 2.2.5. *Take a 0-1 matrix A , considered as an element of M_λ^* . Then $B_\lambda A$ is the (unique) open orbit in some $\epsilon^{-1}(X_{\lambda'})$ if and only if for each $(i, j), (r, s) \in \text{supp } A$ with $i < r, j > s$ we have $(i, s) \in \Delta(\lambda)$.*

Proof. It is not hard to see that $\overline{O_A}$ can be defined by using only conditions on the ranks of submatrices A_ρ where $\rho \subset R_\lambda$ is a rectangle with corners at (k, l) and $(i+1, s+1)$ and where $(i, j), (r, s) \in \text{supp } A$ satisfy $i < r$ and $j > s$. The assumption of this lemma assures that such rectangles touch the boundary of the Young diagram $\Delta(\lambda)$, so the result follows from the characterization of normal slices to Schubert cells in [4]. \square

One case in particular will be important. Say $E = E_{ij}$ is the elementary matrix supported at $(i, j) \in R_\lambda$, and put $(\alpha, \beta) = w^{-1}(i, j)$. The previous lemma shows that $B_\lambda E \subset M_\lambda^*$ is the open orbit in $\epsilon^{-1}(X_{\lambda'})$ for some λ' .

Lemma 2.2.6. *We have $\phi_{\lambda'}(\alpha) = +1$, $\phi_{\lambda'}(\beta) = -1$, and $\phi_\lambda(\gamma) = \phi_{\lambda'}(\gamma)$ for all $\gamma \neq \alpha, \beta$.*

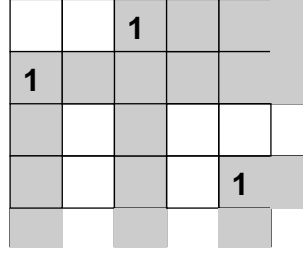
Proof. $\epsilon(0) = W_\lambda$ is spanned by the vectors

$$\{\mathbf{e}_m \mid 1 \leq m \leq n, \text{ and } \phi_\lambda(m - k - \tfrac{1}{2}) = -1\}.$$

A basis for $\epsilon(E)$ can be obtained by replacing $\mathbf{e}_{\alpha+k+\frac{1}{2}}$ by $\mathbf{e}_{\alpha+k+\frac{1}{2}} + \mathbf{e}_{\beta+k+\frac{1}{2}}$. Since $\alpha < \beta$ (Lemma 2.1.2), there is a $g \in B$ for which $g \cdot \epsilon(E) = W_{\lambda'}$ is spanned by the same basis as W_λ with $\mathbf{e}_{\alpha+k+\frac{1}{2}}$ replaced by $\mathbf{e}_{\beta+k+\frac{1}{2}}$. \square

Remark. We will see in §2.5 that if (α, β) is a λ -pair, the dual orbit $(B_\lambda E)^*$ has codimension one in M_λ ; it is the fiber over W_λ of a codimension one orbit of Λ where $\overline{T_{X_\lambda}^* X}$ and $\overline{T_{X_{\lambda'}}^* X}$ intersect.

2.3. Orbit structure of M_λ . In this section we give some general results on the geometry of the B_λ -orbits on M_λ , and the closure relations between them. In the end we will only need to understand the orbits with codimension ≤ 2 , but it will be convenient to study the problem in general first.


 FIGURE 6. The set $\tau(A)$.

Put a partial order on $\mathbb{N} \times \mathbb{N}$ by saying $(i, j) \leq (i', j')$ whenever $i \leq i'$ and $j \leq j'$. Thus w is an order-preserving map, with the order on $\tilde{\Pi}(\lambda)$ from §1.2.

For a 0-1 matrix $A \in M_\lambda$, denote the orbit $B_\lambda A$ by O_A . Define $\tau(A) \subset R_\lambda$ to be the set of points (i, j) for which there is a point $(i', j') \in \text{supp } A$ with $(i', j') \leq (i, j)$ and either $i = i'$ or $j = j'$; i.e., $\tau(A)$ consists of all squares for which there is a 1 either above or to the left; see figure 6.

Proposition 2.3.1. *Suppose $A \in M_\lambda$ is a 0-1 matrix. Then we have*

$$T_A(O_A) = \mathbb{C}^{\tau(A)},$$

where we identify the tangent space to the conical orbit $O_A = B_\lambda \cdot A$ as a subspace of M_λ . In particular, we have $\dim O_A = \#\tau(A)$.

It will be useful to reconstruct the matrix A from $\tau(A)$:

Proposition 2.3.2. *A 0-1 matrix A can be recovered from $\tau(A)$ by the following procedure: begin with $A = 0$, and $S = \tau(A)$. If $S = \emptyset$ then halt; otherwise choose a minimal element (i, j) of S and put $A_{ij} = 1$. Discard all points of S in the same row or column as (i, j) and repeat.*

Given a 0-1 matrix A , we can complete A to an “infinite 0-1 matrix” \hat{A} with entries for all $(i, j) \in \mathbb{N}^2 \setminus \Delta(\lambda)$ by applying this algorithm to the set

$$\tau(A) \cup \{(i, j) \mid i > k \text{ or } j > l\}.$$

It follows that $\hat{A}_{ij} = A_{ij}$ for all $(i, j) \in R_\lambda$. Figure 7 shows an example where $k = l = 3$, $\lambda = \emptyset$.

Since \hat{A} has exactly one 1 in each row and column, it can be thought of as a permutation $\sigma_A: \mathbb{N} \rightarrow \mathbb{N}$: let $\sigma_A(i) = j$ whenever $\hat{A}_{ij} = 1$. Then σ defines an injective map from the set of 0-1 matrices on R_λ to the group $S_\infty = \bigcup S_n$ of permutations of \mathbb{N} which are eventually the identity.

Lemma 2.3.3. *We have*

$$\text{len}(\sigma) = kl - \dim(O_A),$$

where len is the length function on S_∞ considered as a limit of Coxeter groups. Furthermore, we have $O_{A'} \subset \overline{O_A}$ for 0-1 matrices A, A' if and only if the corresponding permutations satisfy $\sigma \leq \sigma'$ in the Bruhat order on S_∞ .

			1		
1					
		1			
	1				
				1	
					1

FIGURE 7. Completing A to \hat{A} .

Proof. The first statement follows since $(i, j) \notin \tau(A) \iff i < \sigma^{-1}(j)$ and $\sigma(i) > j$, whereas $\text{len}(\sigma)$ is the number of pairs (i, j') with $i < j'$ and $\sigma(i) > \sigma(j')$.

For the second statement, we use the following description of the Bruhat order on S_∞ (see [9], page 173). Given $\sigma \in S_\infty$ and $p, q \in \mathbb{N}$,

$$r_\sigma(p, q) = \#\{i \leq p \mid \sigma(i) \leq q\}.$$

Then $\sigma \leq \sigma'$ if and only if $r_\sigma(p, q) \geq r_{\sigma'}(p, q)$ for all p, q . If σ is associated to the matrix \hat{A} , then $r_\sigma(p, q)$ is just the rank of the submatrix $\hat{A}(p, q)$ with corners at $(1, 1)$ and (p, q) , so the “if” part of the lemma is clear.

For the other direction, note that for points $(p, q) \notin R_\lambda$ we can recursively find $r_A(a, b) = \text{rank } \hat{A}(p, q)$: set $r_A(a, 0) = r_A(0, b) = 0$ for all a, b . Then

$$r_A(p, q) = \min(r_A(p-1, q), r_A(p, q-1)) + 1.$$

Thus knowing $r_A(p, q) \leq r_{A'}(p, q)$ holds for $(p, q) \in R_{k,l}$ will imply the same inequality for all (p, q) , giving the “only if” part. \square

Next we describe a procedure which given an orbit, gives a codimension one orbit contained in its closure. Let $A \in M_\lambda$ be a 0-1 matrix, and take a point (i, j) in its support. Take a minimal point $(r, s) > (i, j)$ with $\hat{A}_{rs} = 1$, and define A' by

- $A'_{ij} = A'_{rs} = 0$
- $A'_{is} = A'_{rj} = 1$
- $A'_{tu} = A_{tu}$ if $t \notin \{i, r\}$ or $u \notin \{j, s\}$.

(ignore any points which fall outside of R_λ). In other words, switch the i th and r th rows (or equivalently, the j th and s th columns) of \hat{A} and restrict to R_λ . See Figure 8 for an example.

Proposition 2.3.4. *We have $O_{A'} \subset \overline{O_A}$, $\dim(O_A) = \dim(O_{A'}) + 1$, and all codimension one orbits contained in $\overline{O_A}$ arise this way.*

Proof. The first statement follows from Proposition 2.2.2. To see the second statement, assume first that $(i', j') \in R_\lambda$. Then $\tau(A')$ can be obtained from $\tau(A)$ by interchanging rows i and i' and columns j and j' , and then removing the point

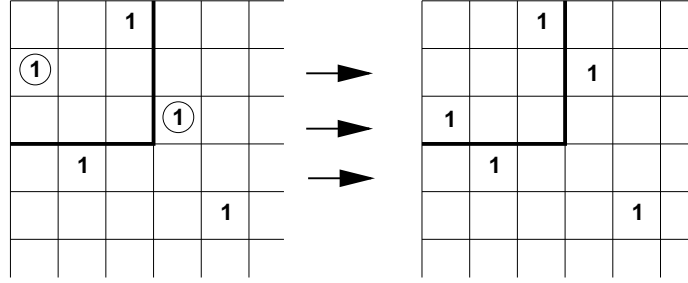


FIGURE 8.

(i, j) . Thus $\#\tau(A') = \#\tau(A) - 1$, as required. If $(i', j') \notin R_\lambda$, we must have either $i' = i + 1$ or $j' = j + 1$ and a similar argument holds.

For the last part, take A, A' so that $O_{A'}$ is a codimension one subvariety of O_A , and let σ, σ' be the corresponding permutations. By Lemma 2.3.3 we have $\sigma \leq \sigma'$ and $\text{len}(\sigma) = \text{len}(\sigma') - 1$. A basic result on reflection groups gives that $\sigma' = \sigma s$, where $s \in S_\infty$ is a reflection. In other words, A' is obtained from A by interchanging two rows and restricting back to R_λ .

Say the i th and r th rows are interchanged, and that $\sigma(i) = j$, $\sigma(r) = s$. Then there is no point (i', j') with $A_{i'j'} = 1$ and $i < i' < r$, $j < j' < s$, since if there were, we would have $\text{len}(\sigma) < \text{len}(\sigma') - 1$. This gives the required minimality of (r, s) . \square

Lemma 2.3.5. *For any non-maximal B_λ orbit $O \subset M_\lambda$, there is an orbit O' with $O \subset \overline{O'}$ and $\dim_{\mathbb{C}} O' = \dim_{\mathbb{C}} O + 1$.*

Proof. Say $O = O_A$, and take a maximal element $(i, j) \in R_\lambda \setminus \text{supp } A$. Suppose $(i, j'), (i', j) \in \text{supp } A$, and define a matrix by $A'_{ij} = A'_{i'j'} = 1$, $A'_{ij'} = A'_{i'j} = 0$, $A'_{rs} = A_{rs}$ for all other (r, s) . Then $O_{A'}$ is the required orbit. \square

2.4. Codimension zero orbits. Let $A_\lambda \in M_\lambda$ be the 0-1 matrix whose support is $w(\Pi_{k,l}(\lambda))$, the image of the set of λ -pairs as defined in §1.2, and let $O_\lambda = O_{A_\lambda}$.

Proposition 2.4.1. *O_λ is the largest B_λ -orbit of M_λ .*

Proof. By Proposition 2.3.1, we need to show that $\tau(A) = R_\lambda$. Take a point $(i, j) \in R_\lambda$. Let $(\alpha, \beta) = w^{-1}(i, j)$. By Lemma 2.1.2, we have $\alpha < \beta$. If (α, β) is a λ -pair, we are done. Otherwise, there are λ -pairs (α, β') and (α', β) . By Lemma 1.2.1, one of these pairs nests inside the other one. Suppose that $\alpha < \alpha' < \beta < \beta'$. Then $(i', j) = w(\alpha', \beta)$ satisfies $A_{i'j} = 1$, and $i' < i$, since $\alpha < \alpha'$. Similarly, the other case gives $A_{ij'} = 1$ with $j' < j$. \square

The matrices A_λ were called “dot configurations” in [4].

Combining Proposition 2.4.1 with the algorithm of Proposition 2.3.2, we obtain:

Lemma 2.4.2. *If $A \in M_\lambda$ is a 0-1 matrix, and (i, j) is a minimal point in $R_\lambda \setminus \tau(A)$, then $(i, j) \in w(\Pi_{k,l}(\lambda))$.*

2.5. Codimension one orbits. Take a λ -pair $\pi = (\alpha, \beta) \in \Pi_{k,l}(\lambda)$, and let $\pi' \in \Pi(\lambda)$ be the parent of π . Letting $(i, j) = w(\pi)$, $(r, s) = w(\pi')$, we can use Proposition 2.3.4 to define a matrix A_λ^π and a corresponding codimension one orbit O_λ^π . Clearly we have

$$\tau(A_\lambda^\pi) = R_\lambda \setminus \{w(\pi)\}.$$

Thus the dual orbit $(O_\lambda^\pi)^*$ is represented by the elementary matrix $E_{w(\pi)}$.

Proposition 2.5.1. *The correspondence $\pi \mapsto O_\lambda^\pi$ gives a bijection from $\Pi_{k,l}(\lambda)$ to the set of codimension one orbits in M_λ .*

We have $\overline{(O_\lambda^\pi)^} = \epsilon^{-1}(X_{\lambda'})$, where $\lambda \xrightarrow{\pi} \lambda'$.*

Proof. The first statement follows from Proposition 2.3.4. For the second, use Lemma 2.2.6. \square

Corollary 2.5.2. *$\lambda \leftrightarrow \lambda'$ if and only if the corresponding components of Λ , $\overline{T_{X_\lambda}^*}$ and $\overline{T_{X_{\lambda'}}^*}$, meet in codimension one.*

2.6. Codimension two orbits.

Theorem 2.6.1. *Let O be a codimension two orbit of M_λ . There are either one or two codimension one orbits whose closures contain O . In either case, they have smooth closures at points of O . If there are two, their closures intersect transversely at points of O .*

Proof. If $O = O_A$, then $N = A + \mathbb{C}^{R_\lambda \setminus \tau(A)}$ gives a normal slice to O at A . It is easy to construct a two-dimensional torus $(\mathbb{C}^*)^2 \subset B_\lambda$ which fixes A , preserves N , and induces the normal crossings stratification on N . Since there must be at least one codimension one orbit O' with $O \subset \overline{O'}$ by Lemma 2.3.5, the stratification on N induced from the orbit stratification of M_λ must be either the normal crossings stratification or the stratification by a complete flag. \square

We will need a combinatorial parametrization of these codimension two orbits. Given $\lambda' > \lambda$ in $\Omega_{k,l}$, let $O_\lambda^{\lambda'} \subset M_\lambda$ be the dual orbit to the open orbit in $\epsilon^{-1}(X_{\lambda'})$.

Proposition 2.6.2. *The map $\lambda' \mapsto O_\lambda^{\lambda'}$ defines a one-to-one correspondence between the set of codimension two orbits $O \subset M_\lambda$ and the set of $\lambda' \in \Omega_{k,l}$ for which there is a diamond (necessarily unique) $(\lambda \rightarrow \lambda_1 \leftrightarrow \lambda' \leftrightarrow \lambda_2 \leftarrow \lambda)$ in Ω where at least one of the λ_m is in $\Omega_{k,l}$. Given such a diamond and orbit, the orbits $O_\lambda^{\lambda_m}$ are exactly the codimension one orbits whose closures contain $O_\lambda^{\lambda'}$.*

Proof. Take a codimension two orbit $O = O_A$. Denote the points in $S = R_\lambda \setminus \tau(A)$ by (i_m, j_m) , $m = 1, 2$. There are two cases:

Case 1: $i_1 \neq i_2$ and $j_1 \neq j_2$. Since A must be obtained from A_λ by two applications of Proposition 2.3.4, we have $(i_m, j_m) = w(\pi_m)$ for $\pi_1, \pi_2 \in \Pi_{k,l}(\lambda)$; neither π_1 or π_2 can be the parent of the other.

The dual orbit O^* contains the 0-1 matrix with support S . We can use Lemma 2.2.5 to get $\lambda' \in \Omega_{k,l}$ so that $\overline{O^*} = \epsilon^{-1}(\overline{X_{\lambda'}})$ – if the points in S are comparable in the partial order on R_λ , this is immediate; otherwise it follows from Proposition

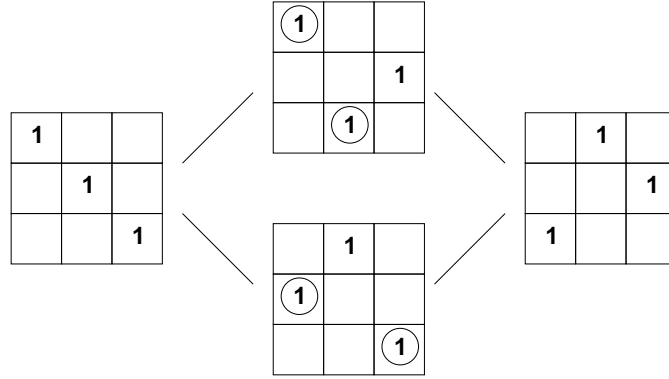


FIGURE 9.

2.3.2. An easy argument along the lines of Lemma 2.2.6 shows that $\lambda_1 \xrightarrow{\pi_2} \lambda'$ and $\lambda_2 \xrightarrow{\pi_1} \lambda'$.

Case 2: The points of S are in the same row or column. The argument is the same in both cases, so assume WLOG that $S = \{(i_1, j_1), (i_2, j_1)\}$ with $i_1 < i_2$. By Lemma 2.4.2 we must have $(i_1, j_1) = w(\pi_1)$ for $\pi_1 \in \Pi_{k,l}(\lambda)$. Since $(i_2, j_1) \notin \Pi(\lambda)$, we must have $w(\pi_2) = (i_2, j_2)$, where $\pi_2 \in \Pi(\lambda)$ is the parent of π_1 (note that π_2 may not be in R_λ).

The dual orbit O^* contains $E_{i_2 j_1}$, so Lemma 2.2.5 gives a partition λ' as before. Lemma 2.2.6 now shows that λ' is the element λ_l in the proof of Lemma 1.8.3.

Finally, the analysis in the proof of Lemma 1.8.3 shows that all diamonds arise either by case 1 or case 2. \square

It is somewhat awkward to index these orbits by diamonds, so we adopt the following notation, based on the classification of diamonds from Lemma 1.8.3. If case 1 holds in the proof of the previous proposition, we write $O = O_\lambda^{\pi_1 \pi_2}$. In case 2, put $O = O_\lambda^{\pi_1, l}$ or $O_\lambda^{\pi_1, r}$ if the points of S are in the same column or row, respectively.

Figure 9 illustrates case 2 with an example: let $k = l = 3$, $\lambda = \emptyset$, and let $\pi_1 = (-\frac{1}{2}, \frac{1}{2})$, so $\pi_2 = (-\frac{3}{2}, \frac{3}{2})$. The rightmost matrix represents the stratum $O_\lambda^{\pi_1, l}$. The corresponding diamond is $\{\emptyset, (1), (2, 1), (1, 1)\}$.

2.7. Fundamental groups of orbits. The stabilizer $(B_\lambda)_A$ of a 0-1 matrix $A \in M_\lambda$ is given by intersecting B_λ with a linear subspace, and is thus connected. As a result, the fundamental group $\pi_1(O_A, A)$ is a quotient of $\pi_1(B_\lambda, 1)$; we can generate $\pi_1(O_A)$ by acting on A by loops in $\pi_1(B_\lambda) \cong \pi_1(T)$, where $T \subset B_\lambda$ is the torus of diagonal matrices. Since these groups are abelian, we can safely ignore basepoints.

For the sake of convenient notation, it is easier to work with the larger torus $\overline{T} \cong (\mathbb{C}^*)^{k+l}$ of all diagonal matrices. Define a collection of generators for $\pi_1(\overline{T})$, indexed by elements $\alpha \in \mathbb{H} \cap (-k, l)$: let

$$\gamma_\alpha(e^{i\theta}) = (1, \dots, e^{i\theta}, \dots, 1),$$

where the $e^{i\theta}$ is in the $(\alpha + k + \frac{1}{2})$ th place.

The element $\gamma_\alpha(z) \in \overline{T}$ acts on M_λ as follows: if $\phi_\lambda(\alpha) = -1$, multiply the row numbered $w_1(\alpha)$ by z . If $\phi_\lambda(\alpha) = +1$, multiply column number $w_2(\alpha)$ by z^{-1} (w_1 and w_2 are the component functions of w ; see §2.1). The action on M_λ^* is the same, with z and z^{-1} interchanged.

It follows that if $O = O_A$ is a B_λ -orbit in M_λ , then $\pi_1(O) \cong \mathbb{Z}^{\#\text{supp } A}$. In particular, if we let $O = O_\lambda$, the rank of $\pi_1(O)$ is $\#\Pi_{k,l}(\lambda)$. We will abuse notation and use the same symbol γ_α to denote a loop in $\pi_1(\overline{T})$ and its image in an orbit O .

Proposition 2.7.1. *The fundamental group $\pi_1(O_\lambda)$ is the abelian group generated by the γ_α , $\alpha \in \mathbb{H}$, with relations $\gamma_\alpha \gamma_\beta = 1$ if $(\alpha, \beta) \in \Pi(\lambda)$, and $\gamma_\alpha = 1$ if $\alpha \notin [-k, l]$.*

Thus a local system on O_λ can be described as a representation of the group algebra $\mathbb{C}\pi_1(O_\lambda)$, which has commuting generators t_α , $\alpha \in \mathbb{H}$ satisfying the relation (2) from the quiver algebra $A_{k,l}$ restricted to V_λ .

Finally, given $O = O_\lambda$, and a codimension 1 orbit $O' = O_\lambda^\pi$ in M_λ , we wish to describe the class in $\pi_1(O)$ of a loop around a point of O' , following the holomorphic orientation of the normal slice to O' . Let $\pi = (\alpha, \beta)$, and let $\pi' = (\alpha', \beta')$ be the parent of π . A normal slice to O' through the 0-1 matrix $A' \in O'$ is given by $N = A' + \mathbb{C}^{w(\pi)}$. Since $\gamma_\alpha(z)\gamma_{\beta'}(z)$ multiplies the entry at $w(\pi)$ by z and fixes A' , this gives the required loop. This calculation will be important in §4 when we explain relation (3) from the quiver algebra $A_{k,l}$.

2.8. Modifications for type D. The analysis of the geometry of $X = X_k^s$ proceeds very similarly to the preceding discussion. We will only indicate the places where the arguments must be changed. Keeping the same definition of $W = W_\lambda$, we have $W_\lambda \in X_k^s$ if and only if $\lambda \in \Omega_k^s$. It is the unique point in X_λ^s which is fixed by the torus $T \subset SO(n, \mathbb{C})$ of diagonal matrices.

2.8.1. Coordinates. The symmetric bilinear form associated to the quadratic form Q gives an identification of \mathbb{C}^n with $(\mathbb{C}^n)^*$, which in turn gives an identification $\mathbb{C}^n/W \cong W^*$ when W is isotropic. Using the identification $T_W X_{k,k} = \text{Hom}(W, \mathbb{C}^n/W) \cong \text{Hom}(W, W^*)$, we have

$$T_W X_k^s = \{\phi \in \text{Hom}(W, W^*) \mid \phi + \phi^t = 0\},$$

so using the standard basis of W and the dual basis of W^* , tangent vectors are represented by skew-symmetric matrices.

Let $R = \{1, \dots, k\} \times \{1, \dots, k\} \setminus \{(i, i) \mid 1 \leq i \leq k\}$. For a subset $S \subset R$ we let $S^t = \{(j, i) \mid (i, j) \in S\}$. If $S = S^t$, define \mathbb{C}_s^S to be the set of skew-symmetric matrices supported on S . We then have $T_W X_\lambda^s = \mathbb{C}_s^{\Delta(\lambda)}$, and so if M_λ is the fiber M_λ of the conormal variety Λ over W_λ we get an identification $M_\lambda \cong \mathbb{C}_s^{R_\lambda}$, where $R_\lambda = R \setminus \Delta(\lambda)$.

Using the standard pairing, the space of skew-symmetric matrices is dual to itself, so we also have an identification $M_\lambda^* \cong \mathbb{C}_s^{R_\lambda}$.

2.8.2. Actions and orbits. If $B_\lambda = B_{W_\lambda}$, then the action of B_λ on M_λ is generated by “symmetric elementary operations”, in which the same operation is performed to both the rows and the columns of a matrix, and rows or columns are only added to

higher-numbered rows or columns. The dual action on M_λ^* is given by truncated versions of these operations, where rows or columns are only added to lower-numbered ones, and anything appearing in $\Delta(\lambda)$ is discarded.

Call a (skew-symmetric) matrix in M_λ or M_λ^* a 0-1 matrix if all of its entries above the diagonal are 0 or 1 and each row or column has at most one nonzero element. Then everything between Proposition 2.2.2 and Proposition 2.2.4 holds word-for-word in type D, including the definition of the embedding $\epsilon: M_\lambda^* \rightarrow X^s$. In particular $A \mapsto O_A = B_\lambda A$ defines a one-to-one correspondence between B_λ orbits in M_λ (or M_λ^*) and 0-1 matrices, as before.

Lemma 2.2.5 must be modified in the type D case:

Lemma 2.8.1. *Take a 0-1 matrix A , considered as an element of M_λ^* . Then $B_\lambda A$ is the (unique) open orbit in some $\epsilon^{-1}(X_{\lambda'})$ if and only if for each $(i, j), (r, s) \in \text{supp } A$ with $i < r, j > s$ and $i \neq s$ we have $(i, s) \in \Delta(\lambda)$.*

Given a point $(i, j) \in R_\lambda$, we define an “elementary matrix” $E_{ij}^s = E_{ij} - E_{ji} \in M_\lambda^*$. Lemma 2.8.1 gives $\lambda' \in \Omega_k^s$ so that $B_\lambda E_{ij}^s$ is the unique open orbit in $\epsilon^{-1}(X_{\lambda'})$. Let $(\alpha, \beta) = w^{-1}(i, j)$. Just as in Lemma 2.2.6, we have:

Lemma 2.8.2. *$\phi_\lambda(\gamma) = \phi_{\lambda'}(\gamma)$ if and only if $\gamma \notin \{\pm\alpha, \pm\beta\}$.*

Define $\tau(A)$ for a 0-1 matrix A exactly as in §2.3 (remember that the diagonal has been removed from R).

Proposition 2.8.3. *If A is a 0-1 matrix, we have*

$$T_A(O_A) = \mathbb{C}_s^{\tau(A)}.$$

Thus $\dim O_A = \frac{1}{2} \# \tau(A)$. The algorithm of Proposition 2.3.2 works verbatim, except that -1 ’s are placed below the diagonal instead of 1s.

As before, we can extend a 0-1 matrix A to a matrix \hat{A} on all of $\mathbb{N} \times \mathbb{N}$, by applying the algorithm of Proposition 2.3.2 to the set

$$\tau(A) \cup \{(i, j) \mid i \neq j \text{ and } (i > k \text{ or } j > k)\}.$$

We can again describe all orbits $O \subset \overline{O_A}$ with $\dim O = \dim O_A - 1$, but the procedure is slightly more complicated than in Proposition 2.3.4. Begin as before with a point $(i, j) \in \text{supp } A$, and assume that $i < j$. Let (i', j') be a minimal point in $\text{supp } \hat{A}$ so that $(i, j) < (i', j')$. If $j < i' < j'$, then switch i' and j' . In geometric terms, this ensures that no corner of the rectangle ρ with corners at (i, j) and (i', j') lies in the reflected rectangle ρ^t . Given these two points, define a 0-1 matrix A' by switching the i th and i' th columns and the i th and i' th rows, multiplying any squares by -1 that are needed to make it a 0-1 matrix.

Proposition 2.8.4. *We have $O_{A'} \subset \overline{O_A}$, $\dim(O_A) = \dim(O_{A'}) + 1$, and all codimension one orbits contained in $\overline{O_A}$ arise this way.*

As an example of the extra step in this procedure, consider the left matrix in Figure 10, representing the largest orbit in M_\emptyset . Let $(i, j) = (1, 2)$ and $(i', j') = (3, 4)$. Then switching the first and third rows and columns (the center matrix) gives a codimension two orbit, while switching the first and fourth columns and rows gives the right matrix, which represents a codimension one orbit.

	1			
-1				
			1	
		-1		

FIGURE 10.

2.8.3. *Codimension zero and one orbits.* Let A_λ be the 0-1 matrix that lies in the open orbit of M_λ , and let $O_\lambda = O_{A_\lambda}$.

Proposition 2.8.5. *We have $\text{supp } A_\lambda = w(\Pi_k^s(\lambda)) \cup [w(\Pi_k^s(\lambda))]^t$.*

Proof. Recalling the definition of $\Pi_k^s(\lambda)$, we see that points in $w(\Pi_k^s(\lambda))$ are of two types: first, all points $(i, j) \in w(\Pi_{k,k}(\lambda))$ with $i < j$, and second, points (i, j) where $(i, i), (j, j) \in \Pi_{k,k}(\lambda)$ run over all points in $w(\Pi_{k,k}(\lambda))$ which lie on the diagonal, taking these points in pairs (the first and second, then the third and fourth, and so on).

It is clear from this description that there is a 0-1 matrix A with the required support, and that $\tau(A) = R_\lambda$. \square

There is a one-to-one correspondence $\pi \mapsto O_\lambda^\pi$ between $\Pi_k^s(\lambda)$ and the set of codimension one orbits in M_λ , defined as follows. Let π' be the parent of π in $\Pi^s(\lambda)$, and set $(i, j) = w(\pi)$, $(i', j') = w(\pi')$. Apply the transformation of Proposition 2.8.4 to the matrix A_λ using these two points; call the resulting matrix A_λ^π , and set $O_\lambda^\pi = O_{A_\lambda^\pi}$.

The description of the dual orbit to O_λ^π given in Proposition 2.5.1 works verbatim.

2.8.4. *Codimension two orbits.* Theorem 2.6.1 remains true in the type D case, with essentially the same proof. Proposition 2.6.2 must be changed slightly, however. Every diamond $(\lambda \rightarrow \lambda_1 \leftrightarrow \lambda' \leftrightarrow \lambda_2 \leftarrow \lambda)$ gives rise to a codimension two orbit

$O = O_\lambda^{\lambda'}$, but there is another kind of codimension two orbit. If $\pi \in \Pi_k^s(\lambda)$, and the parent of π is not in $\Pi(\lambda)$, the two points in $\text{supp } A_\lambda^\pi \setminus \text{supp } A_\lambda$ that lie above the diagonal are parent and child, rather than siblings as in Proposition 2.6.2. This is because the modified rule in Proposition 2.8.4 must be applied.

Applying Proposition 2.8.4 to these two points gives, as before, two codimension two orbits (call them $O_\lambda^{\pi,p}$ and $O_\lambda^{\pi,c}$). The first orbit behaves as in the type A case, but the orbit $O_\lambda^{\pi,c}$ does not correspond to a diamond – O_λ^π is the only codimension one orbit whose closure contains $O_\lambda^{\pi,c}$, even if k is large. The orbits $O_\lambda^{\pi,c}$ are responsible for the extra clause in the quiver relation 4s.

For instance, take $\lambda = \emptyset$. If $\pi_1 = (-\frac{1}{2}, \frac{3}{2})$, $\pi_2 = (-\frac{5}{2}, \frac{7}{2})$, then $O_\lambda^{\pi_1\pi_2,c}$ corresponds to the sequence of partitions $\emptyset \rightarrow (2, 2) \rightarrow (3, 2, 1)$; it is easy to check that this cannot be completed to a diamond.

2.8.5. Fundamental groups. As in the type A case the fundamental group of an orbit O_A is a quotient of $\pi_1(T)$, where $T \subset B_\lambda$ is the torus of diagonal matrices. In terms of the loops γ_α described in §2.7, generators for $\pi_1(T)$ are given by $\bar{\gamma}_\alpha = \gamma_\alpha \gamma_{-\alpha}^{-1}$, for $-k < \alpha < k$.

Proposition 2.8.6. $\pi_1(O_\lambda)$ is the abelian group generated by the $\bar{\gamma}_\alpha$, $\alpha \in \mathbb{H}$, with relations $\bar{\gamma}_\alpha \bar{\gamma}_{-\alpha} = 1$, $\gamma_\alpha \gamma_\beta = 1$ if $(\alpha, \beta) \in \Pi^s(\lambda)$, and $\gamma_\alpha = 1$ if $\alpha \notin [-k, k]$.

In other words, the $\bar{\gamma}_\alpha$ satisfy the relations (2s) for the quiver algebra \mathcal{A}_k^s (§1.7).

The relation (3s) in the quiver describes the class of a loop in the open orbit O_λ around a point of the codimension one orbit O_λ^π in terms of the generators $\bar{\gamma}_\alpha$. Recalling the construction of A_λ^π , we see that $\bar{\gamma}_\alpha \bar{\gamma}_\zeta$ gives the required loop, where $\pi = (\alpha, \beta)$, and ζ is as described in (3s).

3. MICROLOCAL PERVERSE SHEAVES

In this section we collect some facts about microlocal perverse sheaves that we will use in our calculation, along with their local description along conormal varieties that have normal crossings singularities through codimension two.

3.1. Stacks. Our calculation of perverse sheaves rests on the observation that perverse sheaves on a complex manifold X can be seen as microlocal objects, i.e., objects which are locally defined on T^*X . To say this more precisely, we need the language of stacks. We give a sketch of a simplified definition which is sufficient for our application.

A stack on a topological space Y can be thought of intuitively as a “sheaf of categories” on Y . Formally, it consists of a category $\mathcal{S}(U)$ for every open $U \subset Y$, together with a restriction functor $|_V^U: \mathcal{S}(U) \rightarrow \mathcal{S}(V)$ for any open subset $V \subset U$ (when no confusion will arise, we write simply $|_V$). They are required to be compatible in the sense that if $W \subset V \subset U$, then

$$|_W^V \circ |_V^U = |_W^U.$$

Remark. Strictly speaking one cannot say that two functors are equal, so a formally correct definition would include natural isomorphisms relating these two functors, which then must satisfy a sort of cocycle identity. However, for our stacks $\mathcal{S}(U)$ can

be considered as a subcategory of the category of modules over $\mathcal{R}(U)$ for some sheaf of rings \mathcal{R} on Y , and in this case it makes sense to say that the functors are equal. A correct general definition can be found in the Appendix to [11].

In order for \mathcal{S} to be a stack, objects and morphisms must be locally defined. Let $U \subset Y$ be an open set, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a cover of U by open sets. We put $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. Define a category $\mathcal{S}(\mathcal{U})$ by letting objects be collections $\{F_\alpha\}_{\alpha \in A}$ where F_α is an object of $\mathcal{S}(U_\alpha)$, together with isomorphisms

$$\phi_{\alpha\beta}: F_\beta|_{U_{\alpha\beta}} \rightarrow F_\alpha|_{U_{\alpha\beta}}, \alpha, \beta \in A.$$

They are required to satisfy

$$\phi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} \circ \phi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} = \phi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}, \alpha, \beta, \gamma \in A.$$

A morphism $f: (\{F_\alpha\}, \phi_{\alpha\beta}) \rightarrow (\{G_\alpha\}, \psi_{\alpha\beta})$ is a collection of morphisms $f_\alpha: F_\alpha \rightarrow G_\alpha$ satisfying

$$\psi_{\alpha\beta} \circ f_\beta|_{U_{\alpha\beta}} = f_\alpha|_{U_{\alpha\beta}} \circ \phi_{\alpha\beta}$$

for all $\alpha, \beta \in A$.

Definition. We say that \mathcal{S} is a stack if for all U and \mathcal{U} as above, the obvious functor $\mathcal{S}(U) \rightarrow \mathcal{S}(\mathcal{U})$ given by letting $F_\alpha = F|_{U_\alpha}^U$ is an isomorphism of categories.

A *substack* \mathcal{S}' of \mathcal{S} is a stack for which each $\mathcal{S}'(U)$ is a full subcategory of $\mathcal{S}(U)$. We call \mathcal{S} a stack of abelian categories if each $\mathcal{S}(U)$ is abelian, and the restriction functors are exact. In that case, given an object $A \in \mathcal{S}(U)$, we define its support $\text{supp } A \subset U$ to be the smallest closed set for which A restricted to $U \setminus \text{supp } A$ is zero.

Remark. The categories $\mathcal{S}h(X)$ of sheaves and $\mathcal{P}(X)$ of perverse sheaves on X both form stacks of abelian categories on X . The derived category $D^b(X)$, however, is *not* generally a stack. For instance, if $X = \mathbb{CP}^1$, there is a nonzero morphism $\mathbb{Q}_X \rightarrow \mathbb{Q}_X[2]$ whose restriction to any smaller open set is zero.

3.2. The stack \mathcal{E} . Microlocal perverse sheaves form a stack \mathcal{E} of abelian categories on the cotangent bundle T^*X . It has been constructed topologically in [11], or one can use the stack of holonomic regular singularities \mathcal{E}_X -modules, via the Riemann-Hilbert correspondence [1, 13, 14, 15]. It has the following properties:

- If $U \subset X$ is open, there is an equivalence of categories $\mathcal{E}(T^*U) \cong \mathcal{P}(U)$.
- The support of any object in $\mathcal{E}(U)$ is a locally \mathbb{C}^* -invariant Lagrangian variety in U .
- The support of an object $A \in \mathcal{E}(T^*X)$ is equal to its micro-support $SS(A)$, considering it as an object in $\mathcal{P}(X)$.

Given a conical Lagrangian variety $\Lambda \subset T^*X$, we let \mathcal{E}_Λ be the substack of objects supported on Λ . Then we have an equivalence of categories

$$\mathcal{E}_\Lambda(T^*X) \cong \mathcal{P}_\Lambda(X).$$

Thus, if Λ is the conormal variety to a stratification \mathcal{S} of X , then $\mathcal{E}_\Lambda(T^*X)$ is equivalent to the category of \mathcal{S} -constructible perverse sheaves on X .

The following theorem is proved in [11] (also see [14]):

Theorem 3.2.1. *If $U \subset T^*X$ is a Zariski open set and $\dim_{\mathbb{C}}(\Lambda \cap (T^*X \setminus U)) \leq \dim_{\mathbb{C}} \Lambda - 2$, then the restriction functor*

$$\mathcal{P}_{\Lambda}(X) \rightarrow \mathcal{E}_{\Lambda}(U)$$

is a full embedding of categories.

In other words, microlocal perverse sheaves through codimension one have enough information to completely describe perverse sheaves and maps between them, but there may be further relations which must be satisfied in order to extend an object from U to all of T^*X .

Kashiwara has conjectured that microlocal perverse sheaves through codimension two are equivalent to global perverse sheaves. We need the following weaker result:

Theorem 3.2.2. *If $U \subset T^*X$ is a Zariski open set and $\dim_{\mathbb{C}}(\Lambda \cap (T^*X \setminus U)) \leq \dim_{\mathbb{C}} \Lambda - 3$, then*

$$\mathrm{Ext}_{\mathcal{P}_{\Lambda}(X)}^1(A, B) = \mathrm{Ext}_{\mathcal{E}_{\Lambda}(U)}^1(A|_U, B|_U)$$

for any two objects A, B in $\mathcal{P}_{\Lambda}(X)$.

Proof. This follows directly from [15], Theorem 1.2.2. Also see [14], Theorem 10.4.1. \square

3.3. Microlocal Fourier transform. Given a complex vector bundle E , the Fourier transform is a functor $F: D_c^b(E) \rightarrow D_c^b(E^*)$, where $D_c^b(E)$ denotes objects of the derived category which are constructible with respect to some analytic \mathbb{C}^* -conic stratification. We will shift our Fourier transform by $\dim_{\mathbb{C}} E$ from the definition in [16], so that it sends perverse sheaves to perverse sheaves. Let $\Theta: T^*E \cong T^*E^*$ be the natural identification ([14], Proposition 5.5.1).

Proposition 3.3.1. *F preserves the micro-support: $SS(F\mathbf{P}) = \Theta(SS(\mathbf{P}))$. Given $U \subset T^*E$ open and the conormal variety Λ to a conical stratification of E , F induces an isomorphism of categories*

$$\mathcal{E}_{\Lambda}(U) \rightarrow \mathcal{E}_{\Theta(\Lambda)}(\Theta(U)).$$

3.4. Monodromic perverse sheaves. We need to describe carefully the categories of perverse sheaves which give the local descriptions we will glue together. We begin with a model for the codimension one data.

Let L be a complex vector bundle over a smooth connected base B . Let $\mathcal{P}_{\mathrm{mon}}(L)$ be the category of *monodromic* perverse sheaves, i.e. sheaves which are constructible with respect to the stratification (Z, \tilde{L}) , where Z is the zero section, and $\tilde{L} = L \setminus Z$.

We briefly describe the standard model for $\mathcal{P}_{\mathrm{mon}}(L)$. Let L' be the dual vector bundle and let \tilde{L}' be the bundle minus the zero section. We can define an isomorphism of \mathbb{C}^* bundles $\omega: \tilde{L} \rightarrow \tilde{L}'$ by requiring that $\langle \omega(x), x \rangle = 1$ for all x . Given local systems $\mathcal{L}, \mathcal{L}'$ on \tilde{L} and \tilde{L}' respectively, let $\mu: \mathcal{L} \rightarrow \mathcal{L}$ and $\mu': \mathcal{L}' \rightarrow \mathcal{L}'$ be the automorphisms given by following the monodromies around loops given by multiplication by $e^{i\theta}$ and $e^{-i\theta}$, respectively.

Lemma 3.4.1 ([19], [22]). *There is a natural equivalence of categories $R_{\text{mon}}: \mathcal{P}_{\text{mon}}(L) \rightarrow \mathcal{C}$, where \mathcal{C} is the category whose objects are local systems \mathcal{L} and \mathcal{L}' on \tilde{L} and \tilde{L}' respectively, together with morphisms of local systems*

$$p: \omega_* \mathcal{L} \rightarrow \mathcal{L}', \quad q: \mathcal{L}' \rightarrow \omega_* \mathcal{L}$$

satisfying

$$pq + 1 = \mu', \quad \text{and} \quad qp + 1 = \omega_* \mu.$$

If $\mathbf{P} \in \mathcal{P}_{\text{mon}}(L)$, then these local systems are described concretely by

$$\mathcal{L} = \mathcal{L}(\mathbf{P}) = \mathbf{P}|_{\tilde{L}};$$

$$\mathcal{L}' = \mathcal{L}'(\mathbf{P}) = (F\mathbf{P})|_{\tilde{L}'},$$

where $F: \mathcal{P}_{\text{mon}}(L) \rightarrow \mathcal{P}_{\text{mon}}(L')$ is the Fourier transform. The maps p, q are described in [19], [22]. Note that they depend on the choice of a fixed orientation of L .

If L is oriented, let L' have the orientation induced using ω , so if L has the holomorphic orientation, L' has the antiholomorphic one. Then Lemma 3.4.1 gives an equivalence $R'_{\text{mon}}: \mathcal{P}_{\text{mon}}(L') \rightarrow \mathcal{C}'$, where \mathcal{C}' is the category defined above taking L' instead of L . If we define a Fourier transform $F_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}'$ by

$$F_{\mathcal{C}}(\mathcal{L}, \mathcal{L}', p, q) = (\mathcal{L}', \mathcal{L}, \omega^* q, \omega^* p),$$

then there is a natural equivalence $F_{\mathcal{C}} \circ R_{\text{mon}} \cong R'_{\text{mon}} \circ F$.

3.5. Normal crossings perverse sheaves. Let L_1 and L_2 be complex line bundles over a smooth, connected base B , and let $E = L_1 \oplus L_2$. Give E the “normal crossings” stratification $\{S_{00}, S_{01}, S_{10}, S_{11}\}$, where $S_{00} = L_1 \cap L_2$ is the zero section, $S_{11} = E \setminus (L_1 \cup L_2)$, $S_{10} = L_1 \setminus S_{00}$, and $S_{01} = L_2 \setminus S_{00}$. Let $\Lambda = \Lambda_{nc}$ be the corresponding conormal variety, and let $\tilde{\Lambda}_{ij}$ be the component of the smooth locus of Λ lying over S_{ij} .

Set $U^1 = T^*E \setminus \{(x, 0) \in T^*E \mid x \in S_{00}\}$, so U^1 contains the codimension zero and one parts of Λ . By Theorem 3.2.1, $\mathcal{P}_{\Lambda}(E)$ embeds as a full subcategory of $\mathcal{E}_{\Lambda}(U^1)$. We wish to describe the category $\mathcal{E}_{\Lambda}(U^1)$ and the additional relations that objects must satisfy in order to extend to $\mathcal{P}_{\Lambda}(E) = \mathcal{E}_{\Lambda}(T^*E)$.

First we reduce to the case where B is a point. Take $b \in B$, and let $i_b: E_b \rightarrow E$ denote the inclusion of the fiber over b . Give E_b the induced stratification, and define $\Lambda_b, U_b^1 \subset T^*E_b$ as above. Then the restriction functor $i^*[-\dim_{\mathbb{C}} B]: D^b(E) \rightarrow D^b(E_b)$ takes perverse sheaves to perverse sheaves, and microlocalizes to give a functor

$$i_{\mu}^*: \mathcal{E}_{\Lambda}(U^1) \rightarrow \mathcal{E}_{\Lambda_b}(U_b^1).$$

Lemma 3.5.1. *An object $\mathbf{P} \in \mathcal{E}_{\Lambda}(U^1)$ extends to $\mathcal{P}_{\Lambda}(E)$ if and only if $i_{\mu}^* \mathbf{P}$ extends to $\mathcal{P}_{\Lambda_b}(U_b)$.*

Proof. This is easy to see when B is contractible, since the bundles L_1 and L_2 can be trivialized. For the general case, cover B by contractible open sets containing b . Since extensions from $\mathcal{E}_\Lambda(U^1)$ to $\mathcal{E}_\Lambda(T^*E)$ are unique and canonical if they exist, by Theorem 3.2.1, the extensions over these open sets glue together. \square

Now restrict to the case where $E = \{b\}$ is a point, so L_1 and L_2 are just 1-dimensional vector spaces. To avoid naming every map in the quiver diagram, we use the convention that $(v_1 \mid \cdots \mid v_r)$ denotes the composition of the maps along the path with nodes labelled v_1, \dots, v_r .

Proposition 3.5.2. *There is an equivalence of categories $R_{nc}: \mathcal{E}_\Lambda(U^1) \rightarrow \mathcal{C}_{nc}$, where \mathcal{C}_{nc} is the category of representations of the quiver*

$$(1) \quad \begin{array}{ccc} V_{00} & \rightleftarrows & V_{01} \\ \updownarrow & & \updownarrow \\ V_{10} & \rightleftarrows & V_{11} \end{array}$$

which satisfy

$$(00 \mid 01 \mid 00 \mid 10) = (00 \mid 10 \mid 11 \mid 10), \text{ and } (00 \mid 01 \mid 00) \text{ invertible}$$

as well as all relations obtained from these by applying the symmetry group of the square.

The category $\mathcal{P}_\Lambda(E)$ sits inside $\mathcal{E}_\Lambda(U^1)$ as quivers satisfying the additional relations of the form

$$(00 \mid 10 \mid 11) = (00 \mid 01 \mid 11),$$

again applying the symmetries of the square.

Proof. Fix orientations of L_1 and L_2 , and give the dual bundles L'_1 and L'_2 the compatible orientations, as in the last section. Using the description of \mathcal{P}_{mon} from the last section twice, we see that an object $\mathbf{P} \in \mathcal{P}_\Lambda(E \setminus S_{00})$ is described by local systems \mathcal{L}_{ij} on $\tilde{\Lambda}_{ij}$ for $ij = 11, 01, 10$, together with maps (making identifications between these spaces as in Lemma 3.4.1) $\mathcal{L}_{01} \rightleftarrows \mathcal{L}_{11} \rightleftarrows \mathcal{L}_{10}$; these maps must satisfy relations as in Lemma 3.4.1. In particular, the monodromies of \mathcal{L}_{11} are completely determined by what these maps do on stalks of the \mathcal{L}_{ij} .

Applying the Fourier transform and using Proposition 3.3.1, we see that objects in $\mathcal{E}_\Lambda(T^*E \setminus \tilde{\Lambda}_{11})$ are given by local systems and maps $\tilde{\mathcal{L}}_{01} \rightleftarrows \tilde{\mathcal{L}}_{00} \rightleftarrows \tilde{\mathcal{L}}_{10}$, with relations as before.

We can glue these two descriptions using the fact that \mathcal{E} is a stack to get a description of the category $\mathcal{E}_\Lambda(U^1)$; the key fact is that there are natural identifications $\tilde{\mathcal{L}}_{ij}(\mathbf{P}) \cong \mathcal{L}_{ij}(\mathbf{P})$, for $ij = 01$ and 10 . This follows easily from existence of natural isomorphisms ([6], Proposition 2.4) $F \cong F_1 F_2 \cong F_2 F_1$, where F_1 denotes the Fourier transform functors $\mathcal{P}_\Lambda(L_1 \oplus K) \rightleftarrows \mathcal{P}_\Lambda(L'_1 \oplus K)$, $K = L_2$ or L'_2 , and similarly F_2 is the Fourier transform in the second coordinate.

Now fix a basepoint $p_{11} \in \tilde{\Lambda}_{11}$; let $p_{ij} \in \tilde{\Lambda}_{ij}$ be the corresponding points in the other smooth components. The vector space V_{ij} in the quiver (1) is the stalk of \mathcal{L}_{ij} at p_{ij} , and the map $V_{ij} \rightarrow V_{kl}$ is induced from the map $\mathcal{L}_{ij} \rightarrow \mathcal{L}_{kl}$. The maps on

stalks determine the monodromies of all four local systems, and the quiver relations arise because the \mathcal{L}_{ij} are local systems, and the maps are maps of local systems.

The description of the category $\mathcal{P}_\Lambda(E)$ is standard; see [22], [10]. \square

We will need a more microlocal description of the orientation choices we made. Instead of choosing orientations of the L_i , we can give each component of Λ a normal orientation along each codimension one intersection with another component. This gives eight orientations; there are four compatibility relations required to apply Lemma 3.4.1, and there are two further relations needed to make Proposition 3.5.2 work. For instance, normal orientations of Λ_{00} along $\Lambda_{00} \cap \Lambda_{01}$ and of Λ_{10} along $\Lambda_{10} \cap \Lambda_{11}$ both give orientations of L'_2 ; these must agree.

As a consequence of the proof of Proposition 3.5.2, we can also describe the action of the Fourier transform on the quiver (1). If we identify $L_i \cong L'_i$ antiholomorphically, we get functors $F_1, F_2, F: \mathcal{E}_\Lambda(U^1) \rightarrow \mathcal{E}_\Lambda(U^1)$; they are naturally equivalent under R_{nc} to the functors given by reflecting the diagram (1) in a horizontal line, reflecting in a vertical line, and rotating by 180° , respectively.

4. PROOF OF THE MAIN THEOREMS

We now apply the results on microlocal perverse sheaves of §3 to the geometry of §2 to prove our main theorems.

4.1. Reducing the theorems to microlocal geometry. Let X be one of the spaces $X_{k,l}$ or X_k^s , taken with the Schubert stratification and corresponding conormal variety Λ . By the results of §2, Λ has finitely many orbits under the action of B . Recall that there is a one-to-one correspondence between B -orbits of Λ and the union of all B_λ -orbits of M_λ over all λ . Using the notation for the codimension 0, 1, and 2 orbits of M_λ introduced in §2, we let $\mathcal{O}_\lambda^\pi \subset \Lambda$ be the orbit corresponding under Proposition 2.2.1 to $O_\lambda^\pi \subset M_\lambda$, and so on. In particular, $\mathcal{O}_\lambda = \tilde{\Lambda}_\lambda$ is the smooth component of Λ lying over X_λ .

Let U^d be the union of $T^*X \setminus \Lambda$ and all B -orbits in Λ of codimension at most d . The following result explains how the quiver categories from §1.3 and §1.7 arise.

First we show how to deduce our theorems from the following statement.

Theorem 4.1.1. *In the case $X = X_{k,l}$ there is an equivalence of categories*

$$R: \mathcal{E}_\Lambda(U^2) \rightarrow \mathcal{A}_{k,l}.$$

If $X = X_k^s$, there is an equivalence of categories

$$R: \mathcal{E}_\Lambda(U^2) \rightarrow \mathcal{A}_k^s.$$

*In both cases, if \mathbf{P} is in $\mathcal{P}_\Lambda(X) = \mathcal{E}_\Lambda(T^*X)$, then the vector space V_λ from the quiver $R(\mathbf{P}|_{U^2})$ is naturally identified with a stalk of the Morse local system $\mathcal{M}_\lambda \mathbf{P}$.*

The remaining sections of this paper construct the functor R and prove this theorem. The construction uses the description of the geometry of Λ obtained in §2, working inward to describe successively the categories $\mathcal{E}_\Lambda(U^d)$ for $d = 0, 1, 2$.

Proof of main theorems (1.4.1 and 1.7.2). The restriction functor

$$|_{U^2}: \mathcal{P}_\Lambda(X) = \mathcal{E}_\Lambda(T^*X) \rightarrow \mathcal{E}_\Lambda(U^2)$$

is a full embedding of categories, by Theorem 3.2.1. Thus it is enough to show that every isomorphism class of objects of $\mathcal{E}_\Lambda(U^2)$ is in the image of this functor, or in other words that Kashiwara's conjecture holds for our varieties.

We first show that every simple object is in the image. By Theorem 1.8.5 an object $(\{V_\lambda\}, \{t_\alpha\}, \{p(\lambda, \lambda')\})$ in $\mathcal{A}_{k,l}$ or \mathcal{A}_k^s is simple if and only if $\sum_\lambda \dim V_\lambda = 1$. Thus there is one simple object $S_\mu \in \mathcal{A}_{k,l}$ (respectively \mathcal{A}_k^s) for each $\mu \in \Omega_{k,l}$ (resp. Ω_k^s).

We show by induction on $|\mu|$ that $R(\mathbf{IC}^*(\overline{X_\mu})|_{U^2}) \cong S_\mu$. If $|\mu| = 0$, then X_μ is a point and the result is clear. Now, assuming it holds for all ν with $|\nu| < |\mu|$, we need to show that $S = R(\mathbf{IC}^*(\overline{X_\mu})|_{U^2})$ is irreducible; this is enough, since $V_\mu(S)$ is one-dimensional. Suppose S were not irreducible. Then S has a composition series consisting of one copy of S_μ and only using S_ν for $\nu \leq \mu$, and in particular there is a nonzero map between S and some S_ν with $|\nu| < |\mu|$. Using the induction hypothesis, Theorem 4.1.1, and Theorem 3.2.1, we get a nontrivial morphism between $\mathbf{IC}^*(\overline{X_\mu})$ and $\mathbf{IC}^*(\overline{X_\nu})$, a contradiction.

Finally, to complete the proof, use Theorem 3.2.2 and induction to show that any quiver object is in the image of $R \circ |_{U^2}$. \square

4.2. Cutting by a normal slice. We prefer to work on the vector spaces M_λ , M_λ^* and their cotangent bundle(s), rather than on X directly. To that end, define a functor $F_\lambda: \mathcal{P}_\Lambda(X) \rightarrow \mathcal{P}(M_\lambda)$ by

$$F_\lambda \mathbf{P} = F \epsilon_\lambda^* \mathbf{P}[-|\lambda|]$$

(note that $\epsilon_\lambda^* \mathbf{P}$ is conical, so the Fourier transform can be applied).

Recall the splitting $\mathrm{Hom}(\mathbb{C}^*/W_\lambda, W_\lambda) = M_\lambda \oplus \bar{M}_\lambda$ from §2.1. Define a map $\kappa_\lambda: T^*M_\lambda \rightarrow T^*X$ by sending (x, ξ) to $\delta(x, \xi, 0, 0) \in T^*X$, where δ is the inclusion of $T^*M_\lambda \times T^*\bar{M}_\lambda$ into T^*X given by differentiating $\tilde{\epsilon}_\lambda$.

Let $\hat{\Lambda} = \hat{\Lambda}_\lambda = \kappa_\lambda^{-1}(\Lambda)$. Considered as a subset of $T^*M_\lambda^* = T^*M_\lambda$, it is the conormal variety to the stratification of M_λ^* by the sets $\epsilon^{-1}(X_{\lambda'})$, $\lambda' \geq \lambda$. It is contained in the conormal variety to the stratification of M_λ by B_λ orbits, but note that it is *not* in general the conormal variety to a stratification of M_λ .

The next proposition follows easily from the fact that ϵ_λ is transverse to the stratification. Let $\hat{U} \subset T^*M_\lambda$ be an open set, and let $U = \delta(\hat{U} \times T^*\bar{M}_\lambda) \subset T^*X$.

Proposition 4.2.1. *The image of F_λ is contained in $\mathcal{P}_{\hat{\Lambda}}(M_\lambda)$. More generally, F_λ induces an equivalence of categories $\mathcal{E}_\Lambda(U) \rightarrow \mathcal{E}_{\hat{\Lambda}}(\hat{U})$.*

4.3. Codimension zero MPS. Microlocal perverse sheaves on the smooth part of Λ are very easy to describe:

Proposition 4.3.1. *Take $U \subset T^*X$ open with $U \cap \Lambda = \tilde{\Lambda}_\lambda$. Then $\mathcal{E}_\Lambda(U)$ is equivalent to the category of local systems on $\tilde{\Lambda}_\lambda$; one such equivalence is given by the Morse local systems functor $\mathbf{P} \mapsto \mathcal{M}_\lambda \mathbf{P}$.*

A proof is given in [11]. The Morse local system functor is defined in [19].

Since $\tilde{\Lambda}_\lambda \cong \mathcal{O}_\lambda \times X_\lambda$, the Morse local systems of $\mathbf{P} \in \mathcal{P}_\Lambda(X)$ are determined by the following:

Proposition 4.3.2. *There is a natural isomorphism*

$$H^{-kl+|\lambda|}(F_\lambda \mathbf{P})|_{\mathcal{O}_\lambda} \cong (\kappa_\lambda)^*(\mathcal{M}_\lambda \mathbf{P}).$$

4.4. MPS through codimension one. We have seen that all the singularities of our stratifications are conical (see Proposition 2.2.4). Thus we can use the techniques of [6] to describe the category $\mathcal{E}_\Lambda(U^1)$.

Take λ, λ' with $\lambda \xrightarrow{\pi} \lambda'$, and consider the corresponding codimension one orbit $O = O_\lambda^\pi$ in M_λ . Embed its normal bundle $L = L_\lambda^\pi = T_O M_\lambda$ as a tubular neighborhood of O which meets only O and the open orbit \mathcal{O}_λ .

Suppose that $\hat{U} \subset T^*M_\lambda$ contains T^*L , and let $U \subset T^*X$ be the corresponding set as described above. Then define a functor

$$F_\lambda^\pi = F_\lambda|_L: \mathcal{E}_\Lambda(U) \rightarrow \mathcal{P}_{\text{mon}}(L_\lambda^\pi).$$

Applying Proposition 4.2.1 gives immediately

Proposition 4.4.1. *If $\hat{U} = T^*L$, then F_λ^π is an equivalence of categories.*

Following Lemma 3.4.1, a perverse sheaf \mathbf{P} gives rise to local systems

$$\mathcal{L}\mathbf{P} = F_\lambda^\pi \mathbf{P}|_{\tilde{L}}, \quad \mathcal{L}'\mathbf{P} = (FF_\lambda^\pi \mathbf{P})|_{\tilde{L}'},$$

In order to combine the local descriptions of microlocal perverse sheaves from Propositions 4.3.1 and 4.4.1 into a description of $\mathcal{E}_\Lambda(U^1)$, we need to relate $\mathcal{L}\mathbf{P}$ and $\mathcal{L}'\mathbf{P}$ with the Morse local systems $\mathcal{M}_\lambda \mathbf{P}$ and $\mathcal{M}_{\lambda'} \mathbf{P}$. The result obtained in [6] is summarized by the following theorem.

Let $U_\lambda^\pi = (T^*X \setminus \Lambda) \cup \mathcal{O}_\lambda \cup \mathcal{O}_{\lambda'} \cup \mathcal{O}_\lambda^\pi$ be the union of all orbits of Λ containing Λ_λ^π in their closures. Let $\hat{\Lambda}_\lambda^\pi = \mathcal{O}_\lambda^\pi \times \{0\} \subset \hat{\Lambda}$ be the codimension one orbit of $\hat{\Lambda}$ lying over \mathcal{O}_λ^π .

We can embed L and L' as tubular neighborhoods of $\hat{\Lambda}_\lambda^\pi$ in $\kappa_\lambda^{-1}(\mathcal{O}_\lambda \cup \mathcal{O}_\lambda^\pi)$ and $\kappa_{\lambda'}^{-1}(\mathcal{O}_\lambda \cup \mathcal{O}_{\lambda'}^\pi)$, respectively. Define inclusions $i: \tilde{L} \rightarrow \mathcal{O}_\lambda$, $i': \tilde{L}' \rightarrow \mathcal{O}_{\lambda'}$ by restricting these embeddings and then following them by κ_λ .

Theorem 4.4.2. [6] *There is a one-dimensional local system \mathcal{L}_{tw} on \tilde{L}' so that the category $\mathcal{E}_\Lambda(U_\lambda^\pi)$ is equivalent to the category whose objects are triples $(\mathbf{P}_\lambda^\pi, \mathcal{M}, \mathcal{M}')$, where $\mathbf{P}_\lambda^\pi \in \mathcal{P}_{\text{mon}}(L_\lambda^\pi)$, and \mathcal{M} and \mathcal{M}' are local systems on \mathcal{O}_λ and $\mathcal{O}_{\lambda'}$ respectively, together with isomorphisms*

$$\begin{aligned} \mathcal{L}(\mathbf{P}_\lambda^\pi) &\cong i^*(\mathcal{M}), \\ \mathcal{L}'(\mathbf{P}_\lambda^\pi) &\cong (i')^*(\mathcal{M}') \otimes \mathcal{L}_{tw}, \end{aligned}$$

and morphisms are triples of morphisms compatible with these isomorphisms. Under this equivalence of categories, a perverse sheaf \mathbf{P} is sent to the triple $(F_\lambda^\pi \mathbf{P}, \mathcal{M}_\lambda \mathbf{P}, \mathcal{M}_{\lambda'} \mathbf{P})$.

For the spaces we are considering, we have a further simplification:

Proposition 4.4.3. *The local system \mathcal{L}_{tw} is trivial.*

Proof. Let $(\mathbf{P}_\lambda^\pi, \mathcal{M}, \mathcal{M}')$ be the triple corresponding to the simple perverse sheaf $\mathbf{P} = \mathbf{IC}^\bullet(\overline{X_{\lambda'}})$. The Morse local system \mathcal{M}' is trivial, so we need to show that $\mathcal{L}'(\mathbf{P}_\lambda^\pi)$ is also trivial.

There is an action of the fundamental group $\pi_1(T)$ on \mathbf{P} . This action must be trivial, since \mathbf{P} is simple, and the local system $\mathbf{P}|_{X_{\lambda'}}$ is trivial. The torus T acts on L and L' in a manner compatible with the action on the base O . The action of $\pi_1(T)$ on the stalks of $\mathcal{L}'(\mathbf{P}_\lambda^\pi)$ must therefore be trivial.

Since $\pi_1(T)$ generates $\pi_1(O)$, it is enough to show that \mathcal{L}_{tw} has trivial monodromy around a loop in a fiber of the projection $\tilde{L}' \rightarrow O$. But by Lemma 5.10 of [6], this monodromy is $(-1)^{d-1}$, where $d = \dim_{\mathbb{C}} X_{\lambda'} - \dim_{\mathbb{C}} X_\lambda$. The proposition thus follows from the following result, which is a direct consequence of the definition of the relation \leftrightarrow from §1.2. \square

Lemma 4.4.4. *In both Ω and Ω^s , if $\lambda \leftrightarrow \lambda'$, then d is odd.*

Remark. This lemma could also be deduced from [8] (in type A) and [4] (in both types). If it failed to hold for some $\lambda \rightarrow \lambda'$, then the result of [5] would show that $SS(\mathbf{IC}^\bullet(\overline{X_{\lambda'}}))$ contains both $\overline{T_{X_{\lambda'}}^* X}$ and $\overline{T_{X_\lambda}^* X}$, contradicting the fact that it is irreducible.

Let A^1 be the quiver algebra with the same generators and relations as $A_{k,l}$ (or A_k^s in the case $X = X_k^s$), except without the relation 4, and let \mathcal{A}^1 be the category of finite dimensional representations of A^1 .

Proposition 4.4.5. *There is an equivalence of categories $R: \mathcal{E}_\Lambda(U^1) \rightarrow \mathcal{A}^1$.*

Proof. Essentially this is just gluing together the presentations of the categories $\mathcal{E}_\Lambda(U_\lambda^\pi)$ given by Theorem 4.4.2 along the Morse local systems of Proposition 4.3.1.

Given $\mathbf{P} \in \mathcal{P}_\Lambda(X)$, The vector space V_λ in the associated quiver object will be the stalk of the Morse local system $\mathcal{M}_\lambda \mathbf{P}$ at a point $y_\lambda \in \mathcal{O}_\lambda$. The map t_α is the action of the monodromy around the loop $e^{i\theta} \mapsto \gamma_\alpha(e^{i\theta}) \cdot y_\lambda$. Since these loops generate $\pi_1(\tilde{\Lambda}_\lambda)$, these maps completely describe $\mathcal{M}_\lambda \mathbf{P}$. Relations 1 and 2 (resp. 2s) in the quiver descriptions of $\mathcal{A}_{k,l}$ and \mathcal{A}_k^s give the relations among these loops, by Propositions 4.3.1, 2.7.1 and 2.8.6.

The categories $\mathcal{P}_{mon}(L_\lambda^\pi)$ have quiver descriptions using Lemma 3.4.1. We choose the orientation of L_λ^π to be $\eta(\beta) = (-1)^{\beta+\frac{1}{2}}$ times the standard (holomorphic) one, where $\pi = (\alpha, \beta)$. This choice will make the description of the codimension two relations simpler – see §4.6. Lemma 3.4.1 gives maps between the stalks of the local systems $\mathcal{L}(\mathbf{P}_\lambda^\pi)$ and $\mathcal{L}'(\mathbf{P}_\lambda^\pi)$ at points z, z' in L_λ^π and $(L_\lambda^\pi)'$. We can identify these stalks with V_λ and $V_{\lambda'}$, by fixing paths from $i(z)$ to y_λ in \mathcal{O}_λ and from $i'(z')$ to $y_{\lambda'}$ in $\mathcal{O}_{\lambda'}$, and using Theorem 4.4.2. This defines the maps $p(\lambda, \lambda')$ and $p(\lambda', \lambda)$ in the quiver.

The fact that the maps in Lemma 3.4.1 are maps of local systems shows that the maps t_α commute with the maps $p(\lambda, \lambda')$. The quiver relation 3 (resp. 3s) comes from the relations in Lemma 3.4.1, remembering the calculation of a loop around a point of O_λ^π in sections 2.7 and 2.8.5.

This defines the required functor $\mathcal{E}_\Lambda(U^1) \rightarrow \mathcal{A}^1$, and in fact shows that it factors through an equivalence of categories $\mathcal{E}_\Lambda(\mathcal{U}) \rightarrow \mathcal{A}^1$, where \mathcal{U} is the cover of U^1 consisting of U^0 and the sets U_λ^π defined in §4.2. The proposition now follows since \mathcal{E} is a stack. \square

4.5. Codimension two relations. We now show that the relations coming from codimension two strata in Λ give exactly the relation (4) from the quiver categories $\mathcal{A}_{k,l}$ and \mathcal{A}_k^s .

Take $O = O_A \subset M_\lambda$ a codimension two B_λ -orbit. Let O_1, O_2 be the codimension one orbits containing O in their closure (we will assume that there are two; if there is only one the same discussion works with minor modifications). Let $\lambda_1, \lambda_2, \lambda'$ be the corresponding elements of Ω , so that O, O_1 , and O_2 are dual to the open orbits in $\epsilon^{-1}(X_{\lambda'}), \epsilon^{-1}(X_{\lambda_1})$, and $\epsilon^{-1}(X_{\lambda_2})$, respectively. The 4-tuple $(\lambda, \lambda_1, \lambda_2, \lambda')$ forms a diamond, with $\lambda \xrightarrow{\pi_i} \lambda_i, i = 1, 2$ — see Proposition 2.6.2.

Given an object \mathbf{P} in $\mathcal{E}_\Lambda(U^2)$, let \mathbf{P}_O be the restriction of $\nu_O F_\lambda \mathbf{P}[2 + |\lambda| - kl]$ to the fiber $M_\lambda/T_A O \cong \mathbb{C}^{S_O} \cong \mathbb{C}^2$, where $S_O = R_\lambda \setminus \tau(A)$. It is a perverse sheaf in $\mathcal{P}_{\Lambda_{nc}}(\mathbb{C}^{S_O})$, where Λ_{nc} is the conormal variety to the normal crossings stratification, using the decomposition $\mathbb{C}^{S_O} = L_1 \oplus L_2$, $L_i = T_A \overline{O_i}/T_A O$. We also get an induced functor $\mathcal{E}_\Lambda(U^1) \rightarrow \mathcal{E}_{\Lambda_{nc}}(U_E^1)$, where U_E^1 is the set defined in §3.5 which contains all of Λ_{nc} except the codimension two point.

Proposition 4.5.1. *In terms of the quiver descriptions of Proposition 4.4.5, the quiver that Proposition 3.5.2 associates to \mathbf{P}_O for \mathbf{P} in $\mathcal{E}_\Lambda(U^i)$, $i = 1, 2$ is*

$$\begin{array}{ccc} V_{\lambda'} & \rightleftharpoons & V_{\lambda_1} \\ \updownarrow & & \updownarrow \\ V_{\lambda_2} & \rightleftharpoons & V_\lambda \end{array}$$

where the maps are the $p(\cdot, \cdot)$ from the algebra \mathcal{A}^1 .

We prove this result in the remaining sections of the paper.

Proof of Theorem 4.1.1. By Lemma 3.5.1, an object $\mathbf{P} \in \mathcal{E}_\Lambda(U^1)$ extends to an object in $\mathcal{E}_\Lambda(U^2)$ if and only if each \mathbf{P}_O extends to an object in $\mathcal{P}_{\Lambda_{nc}}(\mathbb{C}^{S_O})$; now use Proposition 3.5.2. \square

It is clear from Theorem 4.4.5 that V_λ, V_{λ_1} and V_{λ_2} and the maps between them appear as they do in Proposition 4.5.1. What remains is to explain why $V_{\lambda'}$ and the four maps involving it appear.

4.6. Compatibility of orientations. Our choice of a normal orientation of M_λ along each codimension one stratum O_λ^π produces normal orientations of each irreducible component $\mathcal{O}_\lambda \subset \Lambda$ along the codimension two orbits contained in it. In order to apply Proposition 3.5.2 as we do in the statement of Proposition 4.5.1, we need to show that our choice of orientations (§4.4) is consistent in the sense described after Proposition 3.5.2.

We will show this for type A, and leave the type D case as an exercise. Say that $\pi_i = (\alpha_i, \beta_i)$, and let $\pi'_i = (\alpha'_i, \beta'_i)$ be the parent of π_i in $\Pi(\lambda)$.

First consider the case where π_1 and π_2 are not parent and child in $\Pi(\lambda)$. We then have $\lambda_1 \xrightarrow{\pi_2} \lambda'$ and $\lambda_2 \xrightarrow{\pi_1} \lambda'$. If $\pi'_1 \neq \pi'_2$ then π'_1 is still the parent of π_1 in $\Pi(\lambda_2)$. Thus the orientations around $\mathcal{O}_{\lambda}^{\pi_1}$ and $\mathcal{O}_{\lambda_2}^{\pi_1}$ are both given by multiplication by $(\gamma_{\alpha_1} \gamma_{\beta'_1})^{\eta(\beta_1)}$, and so are compatible.

If $\pi'_1 = \pi'_2$, assume WLOG that $\beta_1 < \alpha_2$. The parent of π_1 in $\Pi(\lambda_2)$ is (α'_2, α_2) . The orientation around $\mathcal{O}_{\lambda_2}^{\pi_1}$ is thus given by multiplication by $(\gamma_{\alpha_1} \gamma_{\alpha_2})^{\eta(\beta_1)}$. Since $\gamma_{\alpha_2} = \gamma_{\alpha'_2}^{-1}$ in $\pi_1(\mathcal{O}_{\lambda_2})$, and $\gamma_{\beta'_1} = \gamma_{\alpha'_2}^{-1}$ in $\pi_1(\mathcal{O}_{\lambda})$, these define compatible orientations. The other compatibility is easier to check, since the parent of π_2 in $\Pi(\lambda_1)$ is always of the form $(\tilde{\alpha}, \beta'_2)$.

Next suppose that π_2 is the parent of π_1 , and let π_3 be the parent of π_1 . As was noted in §2.6 and Lemma 1.8.3, there are two possibilities for λ' . Suppose we have $\lambda_1 \xrightarrow{(\beta_1, \beta_2)} \lambda' = \lambda_r$; the case where $\lambda_1 \xrightarrow{(\alpha_2, \alpha_1)} \lambda'$ is very similar. The orientation around $\mathcal{O}_{\lambda_1}^{(\beta_1, \beta_2)}$ is given by $(\gamma_{\beta_1} \gamma_{\beta_3})^{\eta(\beta_2)}$, and the orientation around $\mathcal{O}_{\lambda}^{\pi_2}$ is given by $(\gamma_{\alpha_2} \gamma_{\beta_3})^{\eta(\beta_2)}$. The same loops are generated by $(\gamma_{\beta_2}^{-1} \gamma_{\beta_3})^{\eta(\beta_2)}$ in both cases. The orientation around $\mathcal{O}_{\lambda}^{\pi_1}$ is given by $(\gamma_{\alpha_1} \gamma_{\beta_2})^{\eta(\beta_1)}$, and the orientation around $\mathcal{O}_{\lambda'}^{(\alpha_2, \alpha_1)}$ is given by $(\gamma_{\alpha_2} \gamma_{\beta_1})^{\eta(\alpha_1)}$. Compatibility follows from the fact that $\beta_1 - \alpha_1$ is odd.

4.7. Relating the functors F_{λ} . To prove Proposition 4.5.1, we need to be able to relate the perverse sheaves $F_{\lambda} \mathbf{P}$ for different λ . Fix $\lambda \in \Omega_{k,l}$, and take an elementary matrix $E = E_{ij}$ in M_{λ}^* . Define $\lambda_0 \in \Omega_{k,l}$ so that $\epsilon(E) \in X_{\lambda_0}$; it is the partition described by Lemma 2.2.6.

Define $S = \{(i, s) \in R_{\lambda} \mid s \leq j\} \cup \{(r, j) \in R_{\lambda} \mid r \leq i\}$; we have $T_E \mathcal{O}_E = \mathbb{C}^S$. If we put $R' = R_{\lambda} \setminus S$, and $Z = \mathbb{C}^{R'}$, we have $Z = (T_E \mathcal{O}_E)^{\perp} \subset M_{\lambda}$. We have a sequence of natural isomorphisms:

$$Z^* \cong M_{\lambda}^* / T_E \mathcal{O}_E = (T_{\mathcal{O}_E} M_{\lambda}^*)|_E \cong (T_{X_{\lambda_0}} X)|_{\epsilon_{\lambda}(E)} \cong (T_{X_{\lambda_0}} X)|_{W_{\lambda_0}} = M_{\lambda_0}^*;$$

the next to last one comes from the trivialization of the normal bundle $T_{X_{\lambda_0}} X$ induced by the coordinate chart $\tilde{\epsilon}_{\lambda_0}$.

Let $\iota: Z \rightarrow M_{\lambda_0}$ be the dual isomorphism. It will be useful to have an explicit description of it. Suppose $(\alpha, \beta) = w_{\lambda}^{-1}(i, j)$ (the map w_{λ} was defined in §2.1).

Lemma 4.7.1. *If $(r, s) = w_{\lambda}(\tilde{\alpha}, \tilde{\beta}) \in R'$, then $\iota(E_{rs}) = E_{r's'}$, where*

$$(r', s') = \begin{cases} w_{\lambda_0}(\tilde{\alpha}, \alpha) & \text{if } \tilde{\beta} = \beta, \\ w_{\lambda_0}(\beta, \tilde{\beta}) & \text{if } \tilde{\alpha} = \alpha, \\ w_{\lambda_0}(\tilde{\alpha}, \tilde{\beta}) & \text{otherwise.} \end{cases}$$

The matrix $E \in M_{\lambda}^*$ gives rise to a linear function $f: M_{\lambda} \rightarrow \mathbb{C}$, given by $f(A) = A_{ij}$. We need the vanishing cycles functor $\phi_f: D^b(M_{\lambda}) \rightarrow D^b(f^{-1}(0))$.

Proposition 4.7.2. *The perverse sheaf $\phi_f(F_{\lambda} \mathbf{P})$ is supported on Z . There is a natural isomorphism*

$$\phi_f(F_{\lambda} \mathbf{P}) \cong \iota^* F_{\lambda_0} \mathbf{P}.$$

Proof. Let $\tilde{Z} = f^{-1}(0) = (\mathbb{C}E)^\perp$, so $\tilde{Z}^* \cong M_\lambda^*/\mathbb{C}E$. Consider the diagram

$$\begin{array}{ccccc}
 D^b(M_\lambda^*) & \xrightarrow{\nu_{\mathbb{C}E}} & D^b(\mathbb{C}E \oplus \tilde{Z}^*) & \xrightarrow{\tilde{h}^*} & D^b(\tilde{Z}^*) \\
 \downarrow & & \downarrow & \searrow & \downarrow \\
 D^b(M_\lambda) & \xrightarrow{\nu_{\tilde{Z}}} & D^b(M_\lambda/\tilde{Z} \oplus \tilde{Z}) & \longrightarrow & D^b(\mathbb{C}E \oplus \tilde{Z}) \xrightarrow{h^*} D^b(\tilde{Z})
 \end{array}$$

Here the leftmost horizontal maps are the indicated specialization functors, $h(v) = (E, v)$ and $\tilde{h}(v^*) = (E, v^*)$ for $v \in \tilde{Z}$, $v^* \in \tilde{Z}^*$, and all the other functors are the appropriate Fourier transforms. There are natural isomorphisms making the left square and the triangle commute by [6], Propositions 2.3 and 2.4. The right quadrilateral commutes by the functoriality of the Fourier transform ([16], Proposition 3.7.13).

The sheaf $\phi_f(F_\lambda \mathbf{P})$ is given by taking $\epsilon_\lambda^*(\mathbf{P}) \in D^b(M_\lambda^*)$ and mapping down and across the bottom to $D^b(\tilde{Z})$. Following the upper path, we get a natural isomorphism

$$\tilde{h}^* \nu_{\mathbb{C}E} \epsilon_\lambda^* \mathbf{P} \cong q^*(\epsilon_{\lambda_0})^* \mathbf{P},$$

where $q: \tilde{Z}^* \rightarrow Z^* \cong M_{\lambda_0}^*$ is the composition of the natural projection and identification maps. The result follows now from [16], Proposition 3.7.14. \square

4.8. Proof of Proposition 4.5.1, continued. We will first complete the proof of Proposition 4.5.1 in the case where the codimension two orbit is $O = O_\lambda^{\pi_1 \pi_2}$, for π_1, π_2 not parent and child. Let $(i_1, j_1) = w(\pi_1)$, $(i_2, j_2) = w(\pi_2)$. We will apply the results of the previous section to the elementary matrix $E = E_{i_1 j_1} \in M_\lambda^*$, so that $\lambda_0 = \lambda_1$.

Define $S \subset R_\lambda$, Z , f , ι as in the last section. Using Lemma 4.7.1, we have $\iota(A_\lambda^{\pi_1}) = A_{\lambda_1}$ and $\iota(A_\lambda^{\pi_1 \pi_2}) = A_{\lambda_1}^{\pi_2}$.

We want to describe the perverse sheaf $\mathbf{P}_f = \phi_f(F_\lambda \mathbf{P})$ near the point $A_\lambda^{\pi_1 \pi_2}$. It will not generally be constructible with respect to the B_λ -orbits, but it will be left invariant by the action of the subgroup $B_{\lambda, E} \subset B_\lambda$ of elements which stabilize $[E] \in \mathbb{P}M_\lambda^*$. Thus it is enough to study \mathbf{P}_f restricted to a normal slice to the orbit $B_{\lambda, E} A_\lambda^{\pi_1 \pi_2}$. One such slice is given by

$$N = A_\lambda^{\pi_1 \pi_2} + \mathbb{C}^{S \cup \{(i_2, j_2)\}}.$$

Using the isomorphism $\mathbf{P}_f \cong \iota^* F_{\lambda_1} \mathbf{P}$ given by Proposition 4.7.2, we see that on $N \cap Z = \mathbb{C}^{\{(i_2, j_2)\}}$, \mathbf{P}_f is given by the quiver $A_{\lambda_1} \rightleftharpoons A_{\lambda'}$, where the maps are $p(\lambda_1, \lambda')$ and $p(\lambda', \lambda_1)$. Thus it will be enough give a natural isomorphism (defining S_O, \mathbf{P}_O as in §4.5)

$$\mathbf{P}_f|_N[i_1 + j_1 + |\lambda| - kl] \cong \phi_h \mathbf{P}_O,$$

where $h: \mathbb{C}^{S_O} \rightarrow \mathbb{C}$ is given by $h(A) = A_{i_1 j_1}$ and we identify $N \cap Z$ with $\mathbb{C}^{\{(i_2, j_2)\}}$ in the obvious way.

By Proposition 2.2.2, the divisor $Y_m = \overline{O_\lambda^{\pi_m}}$, $m = 1, 2$, is given by the vanishing of the determinant of the submatrix with lower right corner at (i_m, j_m) and upper

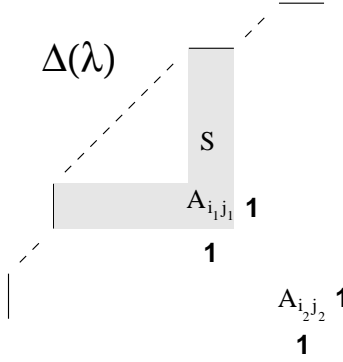


FIGURE 11.

left corner at $w_{\lambda_m}(\pi_m)$. Assume for the moment that $\pi_2 \not\prec \pi_1$. A simple calculation gives

$$Y_1 \cap N = \{A \in N \mid A_{i_1 j_1} + Q(A_{S^-}) = 0\}, \text{ and}$$

$$Y_2 \cap N = \{A \in N \mid A_{i_2 j_2} = 0\}.$$

Here Q is the nondegenerate quadratic form on \mathbb{C}^{S^-} , $S^- = S \setminus \{(i_1, j_1)\}$ given by the formula $Q = \sum A_{ij_1} A_{i_1 j}$, where the sum is over all $(i, j) \in \text{supp } A_{\lambda}^{\pi_1 \pi_2}$ with $(i, j) < (i_1, j_1)$ (see Figure 11).

Applying the change of coordinates

$$\begin{aligned} A'_{i_1 j_1} &= A_{i_1 j_1} + Q(A_{S^-}), \\ A'_{rs} &= A_{rs}, \quad (r, s) \neq (i_1, j_1) \end{aligned}$$

gives $Y_m \cap N = \{A' \in N \mid A'_{i_m j_m} = 0\}$ and $f(A') = A'_{i_1 j_1} - Q(A'_{S^-})$. The claim follows now from the following lemma, which generalizes a result [21] of Sebastiani and Thom; see [20] for a proof.

Lemma 4.8.1. *Take a polynomial $f: \mathbb{C}^n \rightarrow \mathbb{C}$, and let $\tilde{f}: \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}$ be given by*

$$\tilde{f}(\mathbf{v}; x_1, \dots, x_k) = f(\mathbf{v}) + x_1^2 + \dots + x_k^2.$$

Let $j: f^{-1}(0) \rightarrow \tilde{f}^{-1}(0)$ be given by $j(\mathbf{v}) = (\mathbf{v}, 0)$, and let $p_1: \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}^n$ be projection on the first factor. If $\mathbf{A} \in D^b(\mathbb{C}^n)$, there is a natural isomorphism

$$\phi_{\tilde{f}} p_1^* \mathbf{A}[k] \cong Rj_* \phi_f \mathbf{A}.$$

Proposition 4.5.1 now follows; just switch π_1 and π_2 and repeat the argument. If $\pi_2 < \pi_1$, a similar argument works, but the equation for $Y_1 \cap N$ has a slightly different form. We leave the details for the reader. We also omit the type D case; the varieties Y_m have equations given by the vanishing of the appropriate Pfaffians, but otherwise the argument is essentially the same.

4.9. Proof of Proposition 4.5.1, concluded. It only remains to handle the cases $O = O_\lambda^{\pi_1\pi_2,l}$ and $O = O_\lambda^{\pi_1\pi_2,r}$, where $\pi_1, \pi_2 \in \Pi(\lambda)$, and π_2 is the parent of π_1 . The argument is similar for the two cases; we will give it for the first one. Assume WLOG that π_2 is the parent of π_1 , and put $(i_m, j_m) = w_\lambda(\pi_m)$. Apply Proposition 4.7.2, using the elementary matrix $E = E_{i_2j_1}$; we get $\lambda_0 = \lambda'$. Also $\iota(A_\lambda^{\pi_1\pi_2,l}) = A_\lambda^{\pi'}$, where $\pi' = (\beta_1, \beta_2)$ satisfies $\lambda' \xrightarrow{\pi'} \lambda_2$.

Put $\mathbf{P}_f = \phi_f(F_\lambda \mathbf{P}) \cong \iota^* F_{\lambda'} \mathbf{P}$. Defining S, Z as before, we again see that to describe \mathbf{P}_f near $A_\lambda^{\pi_1\pi_2,l}$ it is enough to describe its restriction to the normal slice

$$N = A_\lambda^{\pi_1\pi_2,l} + \mathbb{C}^{S \cup \{(i_2, j_2)\}}.$$

Since \mathbf{P}_f is supported on $Z = \mathbb{C}^{R_\lambda \setminus S}$, $\mathbf{P}_f|_N$ is a monodromic perverse sheaf on $N \cap Z = A_\lambda^{\pi_1\pi_2,l} + \mathbb{C}^{\{(i_2, j_2)\}} \cong \mathbb{C}$. We will show that there is a natural isomorphism

$$\mathbf{P}_f|_N \cong F\phi_h \mathbf{P}_O,$$

letting $h(A) = A_{i_1j_1}$, and identifying $N \cap Z$ with $\mathbb{C}^{\{(i_2, j_2)\}}$ as before.

Again putting $Y_m = \overline{O_\lambda^{\pi_m}}$ for $m = 1, 2$, we get

$$Y_1 \cap N = \{A \in N \mid A_{i_1j_1} = 0\}, \text{ and}$$

$$Y_2 \cap N = \{A \in N \mid A_{i_2j_1} - A_{i_1j_1}A_{i_2j_2} + Q(A_{S^-}) = 0\},$$

where Q is a nondegenerate quadratic form on \mathbb{C}^{S^-} , $S^- = S \setminus \{(i_1, j_1), (i_2, j_2)\}$.

Define a change of variables as follows:

$$\begin{aligned} A'_{i_2j_1} &= A_{i_2j_1} - A_{i_1j_1}A_{i_2j_2} + Q(A_{S^-}), \\ A'_{rs} &= A_{rs}, \quad (r, s) \neq (i_2, j_1). \end{aligned}$$

The result now follows from Lemma 4.8.1 and the following lemma.

Let V, W be one-dimensional complex vector spaces, and take a biconic sheaf $\mathbf{A} \in D^b(V \times W)$, i.e. let \mathbf{A} be constructible with respect to the normal crossings stratification. Fix an element $w^* \in W^*$. Let $p_{12}: V \times W \times V^* \rightarrow V \times W$ be the projection, let $g: V \times W \times V^* \rightarrow \mathbb{C}$ be given by $g(v, w, v^*) = \langle v, v^* \rangle + \langle w, w^* \rangle$. Let $i: V^* \rightarrow V^* \times W^*$ and $j: V^* \rightarrow g^{-1}(0)$ be given by $i(v^*) = (v^*, w^*)$ and $j(v^*) = (0, 0, v^*)$.

Lemma 4.9.1. *There is a natural isomorphism*

$$j^* \phi_g(p_{12}^* \mathbf{A})[1] \cong i^* F \mathbf{A}.$$

Proof. Consider $X = V \times W \times V^*$ as a vector bundle over the base $B = V^*$, so that $E = g^{-1}(0)$ is a sub-vector bundle. Then the vanishing cycles functor ϕ_g is naturally equivalent to $s^* F \nu_E[-1]$, where $\nu_E: X \rightarrow T_E X = E \times_B X/E$ is the specialization functor, F is the Fourier transform $D^b(T_E X) \rightarrow D^b(T_E^* X)$, and $s: E \rightarrow T_E^* X$ is given by $s(\zeta) = (\zeta, dg_\zeta)$.

Now we do a diagram chase similar to the one in Proposition 4.7.2 Consider the following diagram of functors:

$$\begin{array}{ccccccc}
 D^b(V \times W) & \xrightarrow{p_{12}^*} & D^b(X) & \xrightarrow{\nu_E} & D^b(T_E X) & \longrightarrow & D^b(T_E^* X) \xrightarrow{s^*} D^b(E) \\
 \uparrow & & \uparrow & & \uparrow & \nearrow & \nearrow \\
 D^b(V^* \times W^*) & \xrightarrow{p_{12}^*} & D^b(X^*) & \xrightarrow{\nu_{E^\perp}} & D^b(T_{E^\perp} X^*) & \xrightarrow{(s')^*} & D^b(E^*)
 \end{array}$$

where E^\perp is the subbundle of X which annihilates E , the map $s': E^* \rightarrow T_{E^\perp} X^* = E^* \times_B (X/E)^*$ is given by $s'(\xi) = (\xi, dg_{(0, v^*, 0)})$ (putting $v^* = \pi_{E^*}(\xi)$), and all the unmarked arrows are the appropriate Fourier transform functors. Arguing as in Proposition 4.7.2, we see that there is a natural isomorphism

$$j^* \phi_g(p_{12}^* \mathbf{A}) \cong j^* F(s')^* \nu_{E^\perp} p_{12}^* F \mathbf{A}.$$

Now, using the assumption that the vector bundles V and W are one-dimensional, a simple argument shows that there is a natural isomorphism

$$(s')^* \nu_{E^\perp} p_{12}^* F \mathbf{A} \cong (\pi_{E^*})^* i^* F \mathbf{A}.$$

But then for any $\mathbf{C} \in D^b(B)$, $j^* F(\pi_{E^*})^* \mathbf{C}$ is naturally isomorphic to $\mathbf{C}[-1]$, which gives the result. \square

REFERENCES

- [1] E. Andronikov, *A microlocal version of the Riemann-Hilbert correspondence*, Top. Appl. Meth. Nonlin. Anal. 4 (1994), 417–425
- [2] A. Beilinson and J. Bernstein, *Localisation de \mathfrak{g} -modules*, C.R. Acad. Sci. Paris 292 (1981), 15–18.
- [3] A. Beilinson, V. Ginzburg, and W. Sörgel, *Koszul duality patterns in representation theory*, J. Am. Math. Soc.
- [4] B. Boe and J. Fu, *Characteristic cycles associated to Schubert varieties in classical Hermitian symmetric spaces*, Canad. J. Math. **49** (1997), 417–467.
- [5] T. Braden, *On the reducibility of characteristic varieties*, Proc. Amer. Math. Soc., to appear.
- [6] T. Braden and M. Grinberg, *Perverse sheaves on rank stratifications*, Duke Math. J. **96** (1999) 317–362.
- [7] T. Braden and M. Khovanov, in preparation.
- [8] P. Bressler, M. Finkelberg, and V. Lunts, *Vanishing Cycles on Grassmannians*, Duke Math. J. **61** (1990), 763–777.
- [9] W. Fulton, *Young Tableaux*, Cambridge University Press 1997.
- [10] A. Galligo, M. Grainger, and P. Maisonobe, *\mathcal{D} -modules et faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier **35** (1985) 1–48.
- [11] S. Gel'fand, R. MacPherson, and K. Vilonen, *Microlocal perverse sheaves*, in preparation.
- [12] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Springer, 1988.
- [13] M. Kashiwara, *Systems of microdifferential equations*, Birkhauser (1983).
- [14] M. Kashiwara, *Introduction to microlocal analysis*, Monographie 32 de L'Enseignement Math., 1986
- [15] M. Kashiwara and T. Kawai, *On holonomic systems of microdifferential equations III*, Publ. RIMS, Kyoto Univ, **17**, 813–979 (1981)
- [16] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer 1990.
- [17] M. Khovanov, *Functor-valued invariants of tangles*, preprint math.QA/0103190.
- [18] A. Lascoux and P. Schutzenberger, *Polynômes de Kazhdan et Lusztig pour les Grassmanniennes*, Astérisque **87-88** (1981), 249–266.

- [19] R. MacPherson and K. Vilonen, *Elementary construction of perverse sheaves*, Invent. Math. **84** (1986), 403–435.
- [20] D. Massey, *The Sebastiani-Thom isomorphism in the derived category*, Comp. Math. **125** (2001), 353–362.
- [21] M. Sebastiani and R. Thom, *Un Résultat sur la monodromie*, Inv. Math. **13** (1971), 90–96.
- [22] J-L. Verdier, *Prolongement des Faisceaux Pervers Monodromiques*, Astérisque **130** (1985) 218–236.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST,
MA 01003

E-mail address: `braden@math.umass.edu`